Smooth Representations of p-Adic Groups

Claudius Heyer

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Please send all corrections to cheyer@uni-muenster.de

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Chapter 1

Local Fields and Locally Profinite Groups

§1. Non-Archimedean Fields

Recall the archimedean absolute value $|\cdot|_\infty$ on $\mathbb{Q}$ given by $|x|_\infty = x$ if $x \geqslant 0$ and $|x|_\infty = -x$ if $x < 0$. The function $|\cdot|_\infty \colon \mathbb{Q} \to \mathbb{R}_{\geqslant 0}$ satisfies the following properties, where $x, y \in \mathbb{Q}$:

(A1) $|x|_\infty \geqslant 0$, and $|x|_\infty = 0$ if and only if $x = 0$;

(A2) $|xy|_\infty = |x|_\infty \cdot |y|_\infty$;

(A3) $|x+y|_\infty \leqslant |x|_\infty + |y|_\infty$.

How many other ways are there to "measure" rational numbers? Besides the trivial absolute value, defined by $|x|_{\mathrm{triv}} = 1$ if $x \neq 0$ and $|x|_{\mathrm{triv}} = 0$ if $x = 0$, there are many other absolute values which are of number theoretic interest.

We fix a prime number p and measure any integer $x \in \mathbb{Z}$ by the largest power of p that divides x; then x is called "p-adically small" if x is divided by a large power of p. For example, $64 = 2^6$ is 2-adically much smaller than 5. More precisely:

Definition 1.1. For each $x \in \mathbb{Z} \smallsetminus \{0\}$ we put

$$\mathrm{val}_p(x) \coloneqq \max\left\{ i \in \mathbb{Z}_{\geqslant 0} \,\middle|\, p^i \text{ divides } x \text{ in } \mathbb{Z} \right\}.$$

For each $x \in \mathbb{Q} \smallsetminus \{0\}$, we choose $a, b \in \mathbb{Z} \smallsetminus \{0\}$ with $x = \frac{a}{b}$ and put $\mathrm{val}_p(x) \coloneqq \mathrm{val}_p(a) - \mathrm{val}_p(b)$. By convention, we set $\mathrm{val}_p(0) \coloneqq \infty$.

Exercise. Check that $\mathrm{val}_p(x) = \mathrm{val}_p(a) - \mathrm{val}_p(b)$ does not depend on the choice of $a, b \in \mathbb{Z}$ with $x = \frac{a}{b}$. Show that the function $\mathrm{val}_p \colon \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ satisfies the following properties:

- $\mathrm{val}_p(x) = \infty$ if and only if $x = 0$;
- $\mathrm{val}_p(xy) = \mathrm{val}_p(x) + \mathrm{val}_p(y)$ for all $x, y \in \mathbb{Q}$;
- $\mathrm{val}_p(x+y) \geqslant \min\{\mathrm{val}_p(x), \mathrm{val}_p(y)\}$ for all $x, y \in \mathbb{Q}$.

We call the function

$$\begin{aligned} |\cdot|_p \colon \mathbb{Q} &\longrightarrow \mathbb{R}, \\ x &\longmapsto |x|_p \coloneqq p^{-\mathrm{val}_p(x)} \end{aligned}$$

the *p-adic absolute value*.

Exercise. (a) The p-adic absolute value on $\mathbb{Q}$ satisfies the properties (A1), (A2) and

(A3') *Ultrametric triangle inequality:* $|x+y|_p \leqslant \max\{|x|_p, |y|_p\}$, for all $x, y \in \mathbb{Q}$.

Note that (A3') implies (A3).

(b) For each $x \in \mathbb{Q}^\times$ one has $|x|_\infty \cdot \prod_p |x|_p = 1$, where the index in the product runs through all prime numbers.

(c) $|x|_p \leqslant 1$, for all $x \in \mathbb{Z}$. In particular, $|\cdot|_p$ does not satisfy the archimedean property.

As a side note, we mention the following important result:

Ostrowski's Theorem. *Let $|\cdot|$ be a non-trivial absolute value on $\mathbb{Q}$. Then one of the following cases holds true:*

(i) The function $|\cdot|$ is a p-adic absolute value, that is, there exists a prime number p and $\rho \in \mathbb{R}_{>1}$ such that $|x| = \rho^{-\operatorname{val}_p(x)}$ for all $x \in \mathbb{Q}$;

(ii) There exists $\alpha \in \mathbb{R}_{>0}$ such that $|x| = |x|_\infty^\alpha$ for all $x \in \mathbb{Q}$.

Proof. See [Neu13, (3.7) Proposition]. □

The p-adic absolute value on $\mathbb{Q}$ is a special case of a *non-archimedean absolute value*: Let F be a field.

Definition 1.2. A function $|\cdot| \colon F \to \mathbb{R}$ is called a *non-archimedean absolute value* if for all $x, y \in F$ we have:

(NA1) $|x| \geqslant 0$, and $|x| = 0$ if and only if $x = 0$;

(NA2) $|xy| = |x| \cdot |y|$;

(NA3) $|x+y| \leqslant \max\{|x|, |y|\}$ (ultrametric triangle inequality).

The tuple $(F, |\cdot|)$ is called a *non-archimedean field*.

By (NA1) and (NA2), the map $F^\times \to \mathbb{R}_{>0}^\times$, $x \mapsto |x|$, is a group homomorphism. In particular, $|\pm 1| = 1$. We will always assume that $|\cdot|$ is *non-trivial*, that is, there exists $x_0 \in F$ with $|x_0| \neq 0, 1$.

The absolute value $|\cdot|$ endows F with the structure of a topological space: The sets

$$D_{<\varepsilon}(x) := \{y \in F \mid |y - x| < \varepsilon\} \qquad (x \in F,\ \varepsilon \in \mathbb{R}_{>0})$$

form the basis of a topology on F.

Lemma 1.3. *Let $(F, |\cdot|)$ be a non-archimedean field.*

(a) The function $|\cdot| \colon F \to \mathbb{R}$ is continuous.

(b) The functions

$$\begin{aligned} +\colon F \times F &\longrightarrow F, & (x,y) &\longmapsto x+y, \\ \cdot\colon F \times F &\longrightarrow F, & (x,y) &\longmapsto xy, \\ F^\times &\longrightarrow F^\times, & x &\longmapsto x^{-1} \end{aligned}$$

are continuous. In other words, F is a topological field.

Proof. The proof only uses (A3). For $x, y \in F$ we compute

$$|x| = |(x-y)+y| \leqslant |x-y| + |y|.$$

Hence $|x| - |y| \leqslant |x-y|$. From $|y-x| = |x-y|$ we deduce

$$\big||x| - |y|\big|_\infty \leqslant |x-y|,$$

where $|\cdot|_\infty$ is the usual absolute value on $\mathbb{R}$. This shows that $|\cdot|$ is (even Lipschitz) continuous, whence (a).

We now prove (b). Let $x_0, x_1 \in F$ and $\varepsilon > 0$. Pick any $y_i \in D_{<\varepsilon}(x_i)$, for $i = 0, 1$. We compute

$$|(y_0+y_1) - (x_0+x_1)| = |(y_0-x_0) + (y_1-x_1)| \leqslant |y_0-x_0| + |y_1-x_1| < 2\varepsilon,$$

hence $y_0 + y_1 \in D_{<2\varepsilon}(x_0+x_1)$, which shows that addition is continuous. We also have

$$\begin{aligned} |y_0y_1 - x_0x_1| &= |(y_0-x_0)(y_1-x_1) + (y_0-x_0)x_1 + x_0(y_1-x_1)| \\ &< \varepsilon \cdot \big(\varepsilon + |x_0| + |x_1|\big). \end{aligned}$$

Thus, $y_0y_1 \in D_{<\varepsilon(\varepsilon+|x_0|+|x_1|)}(x_0x_1)$, which shows that multiplication is continuous. Finally, let $x \in F^\times$ and $0 < \varepsilon < \frac{|x|}{2}$. For any $y \in D_{<\varepsilon}(x)$ we have $|y| = |x+(y-x)| \geqslant |x| - |y-x| > |x| - \frac{|x|}{2} = \frac{|x|}{2}$. Hence, we have

$$|y^{-1} - x^{-1}| = \left|\frac{x-y}{xy}\right| = \frac{|x-y|}{|x|\cdot|y|} < \frac{2\varepsilon}{|x|^2}.$$

Thus $y^{-1} \in D_{<2\varepsilon|x|^{-2}}(x^{-1})$, which shows that $x \mapsto x^{-1}$ is continuous. □

So far, we have not used the ultrametric triangle inequality. We now study properties which are specific to non-archimedean fields.

Lemma 1.4. *For all $x, y \in F$ one has:*

$$|x| \neq |y| \implies |x+y| = \max\{|x|, |y|\}.$$

Proof. Without loss of generality, we may assume $|x| < |y|$. Then

$$|x| < |y| = |(x+y) - x| \leqslant \max\{|x+y|, |x|\}$$

implies $|y| \leqslant |x+y|$. Conversely, we have $|x+y| \leqslant \max\{|x|, |y|\} = |y|$. □

Lemma 1.5. *Let $(F, |\cdot|)$ be a non-archimedean field.*

(a) The sets $D_{<\varepsilon}(x)$ and $D_{\leqslant\varepsilon}(x) \coloneqq \{y \in F \mid |y-x| \leqslant \varepsilon\}$ are both open and closed in F.

(b) For $x, y \in F$ and $\varepsilon > 0$ with $D_{<\varepsilon}(x) \cap D_{<\varepsilon}(y) \neq \varnothing$ we have $D_{<\varepsilon}(x) = D_{<\varepsilon}(y)$.

(c) F is totally disconnected*, that is, every non-empty connected subset of F is a point.*

Proof. For (a), it is clear that $D_{<\varepsilon}(x)$ is open and $D_{\leqslant\varepsilon}(x)$ is closed. We prove that $D_{<\varepsilon}(x)$ is closed; the fact that $D_{\leqslant\varepsilon}(x)$ is open then follows from a similar argument. Take any $y \in F \smallsetminus D_{<\varepsilon}(x)$. For each $z \in D_{<\varepsilon}(y)$, we have $|z-y| < \varepsilon \leqslant |y-x|$ and hence Lemma 1.4 shows

$$|z-x| = |(z-y)+(y-x)| = \max\{|z-y|, |y-x|\} = |y-x| \geqslant \varepsilon.$$

We conclude $D_{<\varepsilon}(y) \subseteq F \smallsetminus D_{<\varepsilon}(x)$, which shows that $D_{<\varepsilon}(x)$ is closed.

We now prove (b). Fix any $z \in D_{<\varepsilon}(x) \cap D_{<\varepsilon}(y)$. For each $x' \in D_{<\varepsilon}(x)$ we compute

$$|x'-y| = |(x'-x)+(x-z)+(z-y)| \leqslant \max\{|x'-x|, |x-z|, |z-y|\} < \varepsilon.$$

This shows $D_{<\varepsilon}(x) \subseteq D_{<\varepsilon}(y)$. The reverse inclusion follows symmetrically.

It remains to prove (c). Let $M \subseteq F$ be any non-empty connected subset and let $x \in M$. Since M is connected, we have $M \subseteq D_{<\varepsilon}(x)$, because otherwise, $M = \big(M \cap D_{<\varepsilon}(x)\big) \sqcup \big(M \smallsetminus D_{<\varepsilon}(x)\big)$ would be a decomposition into two non-empty open subsets by (a). Hence

$$M \subseteq \bigcap_{\varepsilon>0} D_{<\varepsilon}(x) = \{x\},$$

which shows $M = \{x\}$. □

There is another property of the p-adic absolute value on $\mathbb{Q}$ that we have not considered, yet: The set $|\mathbb{Q}^\times|_p = p^{\mathbb{Z}} \subseteq \mathbb{R}^\times_{>0}$ is discrete.

Definition 1.6. A non-archimedean absolute value $|\cdot|$ on F is called *discrete* if $|F^\times|$ is a discrete subset of $\mathbb{R}^\times_{>0}$. In this case, we call $(F, |\cdot|)$ a *discretely valued* non-archimedean field.

Lemma 1.7. *The absolute value $|\cdot|$ on F is discrete if and only if there exists $r \in \mathbb{R}_{>1}$ such that $|F^\times| = r^{\mathbb{Z}}$.*

Proof. If $|F^\times| = r^{\mathbb{Z}}$ for some $r \in \mathbb{R}_{>1}$, then $|\cdot|$ is clearly discrete. For the converse, it suffices to show that every discrete subgroup $H \neq \{1\}$ of $\mathbb{R}^\times_{>0}$ is of the form $r^{\mathbb{Z}}$ for some $r > 1$. Let $r \in H$ be the smallest element with $r > 1$. Since $\log_r \colon \mathbb{R}^\times_{>0} \to \mathbb{R}$ is a topological isomorphism of groups, it suffices to show that $\mathbb{Z}$ is the unique (non-trivial, discrete) subgroup of $\mathbb{R}$ which contains 1 but no element s with $0 < s < 1$. But this is clear. □

Notation. Suppose that $|\cdot|$ is discrete and let $r \in \mathbb{R}_{>1}$ with $|F^\times| = r^{\mathbb{Z}}$. We denote

$$\operatorname{val}_F \coloneqq -\log_r|\cdot| \colon F^\times \twoheadrightarrow \mathbb{Z}$$

the associated *(normalized) discrete valuation*. We put $\operatorname{val}_F(0) \coloneqq \infty$. Observe that $|\cdot| = r^{-\operatorname{val}_F(\cdot)}$. It satisfies the following properties, for $x, y \in F$:

(V1) $\operatorname{val}_F(x) = \infty$ if and only if $x = 0$;

(V2) $\operatorname{val}_F(xy) = \operatorname{val}_F(x) + \operatorname{val}_F(y)$;

(V3) $\operatorname{val}_F(x+y) \geqslant \min\{\operatorname{val}_F(x), \operatorname{val}_F(y)\}$.

Note that Lemma 1.4 says

$$\operatorname{val}_F(x) \neq \operatorname{val}_F(y) \implies \operatorname{val}_F(x+y) = \min\{\operatorname{val}_F(x), \operatorname{val}_F(y)\}. \tag{1.1}$$

Proposition 1.8. *Let $|\cdot|$ be a (non-trivial) discrete non-archimedean absolute value on F with associated discrete valuation* $\operatorname{val}_F \colon F \twoheadrightarrow \mathbb{Z} \cup \{\infty\}$.

(a) $o_F := \{x \in F \mid |x| \leqslant 1\} = \{x \in F \mid \operatorname{val}_F(x) \geqslant 0\}$ *is a subring of F.*

(b) $\mathfrak{m}_F := \{x \in F \mid |x| < 1\} = \{x \in F \mid \operatorname{val}_F(x) \geqslant 1\}$ *is the unique maximal ideal of o_F. In particular, $(o_F, \mathfrak{m}_F)$ is a local ring, and* $o_F^\times = \{x \in F \mid |x| = 1\} = \{x \in F \mid \operatorname{val}_F(x) = 0\}$.

(c) *o_F is a principal ideal domain.*

(d) *Any $\varpi \in o_F$ with $\operatorname{val}_F(\varpi) = 1$ generates $\mathfrak{m}_F$ and is a prime element.*

Proof. It is clear from (NA2) (or (V2)) that o_F and $\mathfrak{m}_F$ are closed under multiplication with o_F. It follows from the ultrametric triangle inequality (NA3) (or (1.1)) that $\mathfrak{m}_F$ and o_F are closed under addition. Hence, o_F is a subring of F and $\mathfrak{m}_F$ is an ideal of o_F. For any $x \in o_F \smallsetminus \mathfrak{m}_F$ we have $\operatorname{val}_F(x^{-1}) = -\operatorname{val}_F(x) = 0$ and hence $x^{-1} \in o_F$. This shows $o_F \smallsetminus \mathfrak{m}_F \subseteq o_F^\times$. It follows from $1 \notin \mathfrak{m}_F$ that $\mathfrak{m}_F$ is a proper ideal of o_F and hence $o_F^\times \subseteq o_F \smallsetminus \mathfrak{m}_F$. We deduce

$$o_F \smallsetminus \mathfrak{m}_F \overset{\text{def.}}{=} \{x \in F \mid \operatorname{val}_F(x) = 0\} = o_F^\times.$$

In particular, $\mathfrak{m}_F$ is the unique maximal ideal in o_F. It is clear that o_F is an integral domain. Let $\mathfrak{a}$ be a non-zero ideal in o_F. There exists $a \in \mathfrak{a}$ with

$$\operatorname{val}_F(a) = \min_{a' \in \mathfrak{a}} \operatorname{val}_F(a') < \infty.$$

It is clear that $(a) \subseteq \mathfrak{a}$. Conversely, let $a' \in \mathfrak{a}$. Then $\operatorname{val}_F(\frac{a'}{a}) = \operatorname{val}_F(a') - \operatorname{val}_F(a) \geqslant 0$ and hence $\frac{a'}{a} \in o_F$. Therefore, $a' = \frac{a'}{a} \cdot a \in (a)$, and this proves $\mathfrak{a} = (a)$. Hence, o_F is a principal ideal domain. The argument also shows that any $\varpi \in o_F$ with $\operatorname{val}_F(\varpi) = 1$ generates $\mathfrak{m}_F$. It remains to show that ϖ is a prime element. But this follows immediately from the fact that $\mathfrak{m}_F = (\varpi)$ is a prime ideal. □

Definition 1.9. The ring o_F of Proposition 1.8 is called the *valuation ring* of F. Any generator ϖ of $\mathfrak{m}_F$ is called a *uniformizer*. The field

$$\kappa_F := o_F/\mathfrak{m}_F$$

is called the *residue field* of F.

Example 1.10. The valuation ring of $(\mathbb{Q}, |\cdot|_p)$ is

$$\mathbb{Z}_{(p)} := \left\{ x \in \mathbb{Q} \,\middle|\, x = \frac{a}{b} \text{ with } a, b \in \mathbb{Z} \text{ and } p \nmid b \right\}$$

with uniformizer p and maximal ideal $p\mathbb{Z}_{(p)}$. The inclusion $\mathbb{Z}/p\mathbb{Z} \hookrightarrow \mathbb{Z}_{(p)}/p\mathbb{Z}_{(p)}$ is surjective: If $\frac{a}{b} \in \mathbb{Z}_{(p)}$ with $p \nmid b$, there exist $m, n \in \mathbb{Z}$ with $a = bm + pn$ and hence $\frac{a}{b} = m + p\frac{n}{b} \equiv m \mod p\mathbb{Z}_{(p)}$. We conclude that the residue field of $\mathbb{Z}_{(p)}$ is $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

§2. Completion

The property of $\mathbb{R}$ that allows one to do analysis is its completeness. The Intermediate Value Theorem applies to show that every real polynomial $f(t) \in \mathbb{R}[t]$ of odd degree has a root in $\mathbb{R}$. Moreover, if $f(t) \in \mathbb{R}[t]$ has a real root r, the Newton method can be used to construct a sequence $(r_n)_n$ in $\mathbb{R}$ with $\lim_{n\to\infty} r_n = r$. In view of its applications to solving Diophantine equations (that is, finding roots of polynomials with coefficients in $\mathbb{Z}$), one would like to consider non-archimedean fields which are complete.

Let $(F, |\cdot|)$ be a non-archimedean field. We recall the following notions:

Definition 2.1. A sequence $(x_n)_n$ in F is called:

(a) *convergent* if there exists $x \in F$ such that for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{Z}_{>0}$ such that $x_n \in D_{<\varepsilon}(x)$ for all $n \geqslant n_0$.

(b) a *Cauchy sequence* if for all $\varepsilon > 0$ there exists $n_0 \in \mathbb{Z}_{>0}$ such that $|x_m - x_n| < \varepsilon$ for all $m, n \geqslant n_0$.

As usual, we have:

- If $(x_n)_n$ converges to $x \in F$, then x is uniquely determined and is called the *limit* of the sequence $(x_n)_n$; we write $x =: \lim_{n\to\infty} x_n$.
- Every convergent sequence is a Cauchy sequence.
- Every Cauchy sequence is bounded.

As a consequence of the ultrametric triangle inequality, we have

- $(x_n)_n$ is a Cauchy sequence if and only if $|x_{n+1} - x_n| \to 0$ for $n \to \infty$.

Definition 2.2. The field $(F, |\cdot|)$ is called *complete* if every Cauchy sequence converges.

Theorem 2.3. *Let $(F, |\cdot|)$ be a non-archimedean field. Up to isometric isomorphism, there exists a unique complete non-archimedean field $(\widehat{F}, \|\cdot\|)$ satisfying:*

(i) $F \subseteq \widehat{F}$ and $\|\cdot\|_{|F} = |\cdot|$.

(ii) F is dense in $\widehat{F}$.

We call $\widehat{F}$ the completion *of F with respect to $|\cdot|$.*

Proof. We first prove uniqueness: Let $(\widehat{F}_i, \|\cdot\|_i)$, for $i = 1, 2$, be two completions of $(F, |\cdot|)$ and denote $\iota_i \colon F \hookrightarrow \widehat{F}_i$ the corresponding embedding of fields. We define a map

$$\varphi \colon (\widehat{F}_1, \|\cdot\|_1) \longrightarrow (\widehat{F}_2, \|\cdot\|_2)$$

as follows: Since $F \subseteq \widehat{F}_1$ is dense, we may choose for any $x \in \widehat{F}_1$ a sequence $(x_n)_n$ in F such that $\iota_1(x_n) \to x$ for $n \to \infty$. If $(x'_n)_n$ is another sequence in F with $\lim_{n\to\infty} \iota_1(x'_n) = x$, then

$$\begin{aligned}\|\iota_2(x'_n) - \iota_2(x_n)\|_2 &= \|\iota_2(x'_n - x_n)\|_2 = |x'_n - x_n| \\ &= \|\iota_1(x'_n - x_n)\|_1 = \|\iota_1(x'_n) - \iota_1(x_n)\|_1 \xrightarrow{n\to\infty} 0.\end{aligned}$$

Hence, the definition $\varphi(x) := \lim_{n\to\infty} \iota_2(x_n)$ is independent of $(x_n)_n$. It is trivial to check that $\varphi\colon \widehat{F}_1 \to \widehat{F}_2$ is a homomorphism of fields and satisfies $\|\varphi(x)\|_2 = \|x\|_1$ for all $x \in \widehat{F}_1$. By interchanging the roles of $\widehat{F}_1$ and $\widehat{F}_2$, we obtain an isometry $\psi\colon (\widehat{F}_2, \|\cdot\|_2) \to (\widehat{F}_1, \|\cdot\|_1)$ of fields. Unraveling the definitions, it is clear that $\varphi \circ \psi = \mathrm{id}_{\widehat{F}_2}$ and $\psi \circ \varphi = \mathrm{id}_{\widehat{F}_1}$. Hence, φ is an isometric isomorphism.

We now prove the existence statement. Let $\mathcal{C}$ be the set of all Cauchy sequences in F. The componentwise operations

$$(x_n)_n + (y_n)_n := (x_n + y_n)_n \qquad \text{and} \qquad (x_n)_n \cdot (y_n)_n := (x_n y_n)_n$$

define on $\mathcal{C}$ the structure of a commutative ring: The only claim that is not immediately clear is that $(x_n y_n)_n$ is a Cauchy sequence if $(x_n)_n$ and $(y_n)_n$ are. As the sequences $(x_n)_n$ and $(y_n)_n$ are bounded, we find $C \in \mathbb{R}_{>0}$ such that $|x_n|, |y_n| \leqslant C$ for all n. Let now $\varepsilon > 0$ and choose n_0 such that $|x_m - x_n|, |y_m - y_n| < \frac{\varepsilon}{2C}$ for all $m, n \geqslant n_0$. Then

$$\begin{aligned} |x_m y_m - x_n y_n| &= |x_m \cdot (y_m - y_n) + (x_m - x_n) y_n| \leqslant |x_m| \cdot |y_m - y_n| + |x_m - x_n| \cdot |y_n| \\ &< C \cdot \frac{\varepsilon}{2C} + \frac{\varepsilon}{2C} \cdot C = \varepsilon \end{aligned}$$

for all $n, m \geqslant n_0$. Hence, $(x_n y_n)_n \in \mathcal{C}$. The map $F \to \mathcal{C}$, $x \mapsto (x, x, x, \dots)$, is clearly a ring homomorphism. Let $\mathcal{N} \subseteq \mathcal{C}$ be the subset of all sequences which converge to zero. It is clearly closed under addition. Since every Cauchy sequence is bounded, $\mathcal{N}$ is also closed under multiplication with elements of $\mathcal{C}$. In other words, $\mathcal{N} \subseteq \mathcal{C}$ is an ideal. We claim that

$$\widehat{F} := \mathcal{C}/\mathcal{N}$$

is a field. Let $x \in \widehat{F} \smallsetminus \{0\}$ which is represented by a Cauchy sequence $(x_n)_n$. Then only finitely many of the x_n's are zero and hence, after replacing $(x_n)_n$ by a different representative if necessary, we may assume $x_n \neq 0$ for all n. Note that there exists $c > 0$ such that $|x_n| \geqslant c$ for all n, because otherwise we could construct a subsequence of $(x_n)_n$ converging to zero, which implies $x = 0$. Now, $|x_{n+1}^{-1} - x_n^{-1}| = \frac{|x_n - x_{n+1}|}{|x_n| \cdot |x_{n+1}|} \leqslant c^{-2} |x_n - x_{n+1}| \to 0$ for $n \to \infty$, which shows that $(x_n^{-1})_n$ is a Cauchy sequence. Hence, $y = (x_n^{-1})_n + \mathcal{N} \in \widehat{F}$ defines the inverse of x. Thus, $\widehat{F}$ is a field and the composite $\iota\colon F \hookrightarrow \mathcal{C} \twoheadrightarrow \widehat{F}$ is a field embedding. For each $x \in \widehat{F}$, we put

$$\|x\| := \lim_{n\to\infty} |x_n|, \tag{1.2}$$

where $(x_n)_n$ is any Cauchy sequence representing x. One checks that this definition does not depend on the choice of $(x_n)_n$ and that $\|\cdot\|$ is a non-archimedean absolute value on $\widehat{F}$. It is clear from the construction that $\iota\colon (F, |\cdot|) \to (\widehat{F}, \|\cdot\|)$ is an isometric embedding and that $\iota(F)$ is dense in $\widehat{F}$. □

Remark. If $(F, |\cdot|)$ is discretely valued with completion $(\widehat{F}, \|\cdot\|)$, then (1.2) shows $\|\widehat{F}^\times\| = |F^\times|$. In other words, $\|\cdot\|$ is also discrete.

Example 2.4. The completion of $(\mathbb{Q}, |\cdot|_p)$ is denoted $\mathbb{Q}_p$ and called the *field of p-adic numbers*. The extension of $|\cdot|_p$ to $\mathbb{Q}_p$ is again denoted $|\cdot|_p$. The valuation ring

$$\mathbb{Z}_p := \{x \in \mathbb{Q}_p \mid |x|_p \leqslant 1\}$$

is called the *ring of p-adic integers*. Since $|\mathbb{Q}_p^\times|_p = |\mathbb{Q}^\times|_p = p^{\mathbb{Z}}$, it follows from Proposition 1.8 that $\mathbb{Z}_p$ is a local principal ideal domain with uniformizer p and maximal ideal $p\mathbb{Z}_p$. The residue field of $\mathbb{Q}_p$ is $\mathbb{F}_p$. More generally, we have:

Lemma 2.5. *Let $(F, |\cdot|)$ be a non-archimedean field with completion $(\widehat{F}, \|\cdot\|)$. Then o_F is dense in $o_{\widehat{F}}$ and $\kappa_{\widehat{F}} \cong \kappa_F$.*

Proof. The kernel of the composite $o_F \to o_{\widehat{F}} \twoheadrightarrow o_{\widehat{F}}/\mathfrak{m}_{\widehat{F}} = \kappa_{\widehat{F}}$ is $o_F \cap \mathfrak{m}_{\widehat{F}} = \mathfrak{m}_F$. This induces an inclusion $\kappa_F \hookrightarrow \kappa_{\widehat{F}}$. Since F is dense in $\widehat{F}$, we may choose for any $x \in o_{\widehat{F}}$ some $y \in F$ with $\|y - x\| < 1$. Hence $y - x \in \mathfrak{m}_{\widehat{F}}$. Then $y = x + (y - x) \in F \cap o_{\widehat{F}} = o_F$, and $y + \mathfrak{m}_F$ is a preimage of $x + \mathfrak{m}_{\widehat{F}}$. □

Exercise. If $(F, |\cdot|)$ in the above lemma is discretely valued, then $o_F/\mathfrak{m}_F^n \cong o_{\widehat{F}}/\mathfrak{m}_{\widehat{F}}^n$, for all $n \in \mathbb{Z}_{\geqslant 0}$.

§3. Local Fields

Definition 3.1. A *local field* is a complete, discretely valued non-archimedean field F with finite residue field κ_F.

Lemma 3.2. *Let F be a local field and $\varpi \in o_F$ a uniformizer. For every element $x \in F$ there exists a unique $n \in \mathbb{Z}$ and $x_0 \in o_F^\times$ such that $x = x_0\varpi^n$. The integer $n = \operatorname{val}_F(x)$ is independent of the choice of ϖ. In other words, one has $F^\times = \varpi^{\mathbb{Z}} \cdot o_F^\times \cong \mathbb{Z} \times o_F^\times$.*

Proof. For $n \coloneqq \operatorname{val}_F(x)$ we have $\operatorname{val}_F(x\varpi^{-n}) = \operatorname{val}_F(x) - n\operatorname{val}_F(\varpi) = 0$ and hence $x_0 \coloneqq x\varpi^{-n} \in o_F^\times$. It is clear that x_0 and n are unique with $x = x_0\varpi^n$. □

Proposition 3.3. *Let F be a local field with uniformizer ϖ. Let $R \subseteq o_F$ be a subset with $0 \in R$ and such that the composite map $R \subseteq o_F \twoheadrightarrow \kappa_F$ is bijective. Any series*

$$x = \sum_{i \geqslant n_0} a_i \varpi^i, \tag{1.3}$$

where $a_i \in R$ and $n_0 \in \mathbb{Z}$ is fixed, converges in F, and each $x \in F$ can be written uniquely in this form. Moreover, $\operatorname{val}_F(x) = n_0$ if $a_{n_0} \neq 0$.

Proof. The partial sums $x_n \coloneqq \sum_{i=n_0}^n a_i\varpi^i$ satisfy

$$|x_{n+1} - x_n| = |a_{n+1}| \cdot |\varpi|^{n+1} \xrightarrow{n\to\infty} 0.$$

Thus, $(x_n)_n$ is a Cauchy sequence in F and hence converges to a unique element in F by completeness.

Let now $x \in F$ and let $n_0 = \operatorname{val}_F(x) \in \mathbb{Z}$. Replacing x with $\varpi^{-n_0}x$, we may assume $x \in o_F$. We inductively construct a sequence $(a_i)_i$ in R such that

$$x \equiv \sum_{i=0}^n a_i\varpi^i \mod \mathfrak{m}_F^{n+1}, \tag{1.4}$$

for all $n \geqslant -1$. Assume $a_0, \dots, a_n \in R$ are constructed such that (1.4) holds (for $n = -1$ this is vacuous). Then $z \coloneqq \varpi^{-n-1} \cdot (x - \sum_{i=0}^n a_i\varpi^i) \in o_F$, and we find a unique $a_{n+1} \in R$ such that $z \equiv a_{n+1} \mod \mathfrak{m}_F$. It follows that $x \equiv \sum_{i=0}^{n+1} a_i\varpi^i \mod \mathfrak{m}_F^{n+2}$. We have thus constructed $(a_i)_i$ in R such that $x - \sum_{i\geqslant 0} a_i\varpi^i \in \mathfrak{m}_F^n$ for all $n \geqslant 0$. Since $\bigcap_{n\geqslant 0} \mathfrak{m}_F^n = \{0\}$, we deduce $x = \sum_{i\geqslant 0} a_i\varpi^i$. □

Example 3.4. Every element $x \in \mathbb{Q}_p^\times$ admits a unique p-adic expansion

$$x = \sum_{i \geqslant n_0} a_i p^i,$$

where $a_i \in \{0, 1, \ldots, p-1\}$ and $n_0 \in \mathbb{Z}$ with $a_{n_0} \neq 0$. Moreover, $x \in \mathbb{Z}_p$ if and only if $n_0 \geqslant 0$, and $x \in \mathbb{Z}_p^\times$ if and only if $n_0 = 0$ (and $a_0 \neq 0$).

Corollary 3.5. *Let F be a local field. Then o_F is compact. In particular, F is locally compact.*

Proof. Let $R \subseteq o_F$ be as in Proposition 3.3. Assume for a contradiction that o_F is not compact, and let $o_F = \bigcup_{\lambda \in \Lambda} U_\lambda$ be an open covering which has no finite subcovering. We construct a sequence $(a_n)_n$ in R such that $\sum_{i=0}^n a_i \varpi^i + \varpi^{n+1} o_F$ is not covered by finitely many U_λ's, for all $n \geqslant 0$. Assume we have already constructed $a_0, \ldots, a_n$. Then

$$\sum_{i=0}^n a_i \varpi^i + \varpi^{n+1} o_F = \sum_{i=0}^n a_i \varpi^i + \varpi^{n+1} \Big(\bigcup_{a \in R} a + \varpi o_F \Big) = \bigcup_{a \in R} \Big(\sum_{i=0}^n a_i \varpi^i + a \varpi^{n+1} \Big) + \varpi^{n+2} o_F$$

is an open covering. As R is finite, there exists $a_{n+1} \in R$ such that $\sum_{i=0}^{n+1} a_i \varpi^i + \varpi^{n+2} o_F$ cannot be covered by finitely many U_λ's. This finishes the construction of $(a_n)_n$ with the desired property.

The sequence $(\sum_{i=0}^n a_i \varpi^i)_n$ converges by Proposition 3.3 to an element $x := \sum_{i=0}^\infty a_i \varpi^i \in o_F$. Choose $\lambda_0 \in \Lambda$ such that $x \in U_{\lambda_0}$. But then we find $n \gg 0$ such that $\sum_{i=0}^{n-1} a_i \varpi^i + \mathfrak{m}_F^n = x + \mathfrak{m}_F^n \subseteq U_{\lambda_0}$, a contradiction to the fact that $\sum_{i=0}^{n-1} a_i \varpi^i + \mathfrak{m}_F^n$ cannot be covered by finitely many U_λ's. ☐

Exercise (Teichmüller representatives). Let F be a local field with residue field $\kappa_F = o_F / \mathfrak{m}_F$ of characteristic $p > 0$. Fix a uniformizer ϖ.

(a) Let $a, b \in o_F$ and $m \in \mathbb{Z}_{\geqslant 1}$ such that $a \equiv b \mod \mathfrak{m}_F^m$. Show $a^{p^n} \equiv b^{p^n} \mod \mathfrak{m}_F^{n+m}$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

(b) Recall that the residue field κ_F is *perfect*, which means that the map $x \mapsto x^p$ is bijective. Let $z \in \kappa_F$ and choose $x_n \in o_F$ such that $(x_n + \mathfrak{m}_F) = z^{p^{-n}}$ for all $n \in \mathbb{Z}_{\geqslant 0}$.

  (i) Show that $(x_n^{p^n})_n$ is a Cauchy sequence in o_F and hence converges to a unique element $[z] \in o_F$.

  (ii) Show that $[z] = \lim_{n \to \infty} x_n^{p^n}$ is independent of the choice of the sequence $(x_n)_n$.

  (iii) Show that the map $[\cdot] \colon \kappa_F \to o_F$ satisfies $[zw] = [z] \cdot [w]$ and $[1] = 1$.

(c) Conclude that o_F contains all $(\#\kappa_F - 1)$-th roots of unity. (For example, $\mathbb{Z}_p$ contains all $(p-1)$-th roots of unity.)

§4. Locally Profinite Groups

We have seen that each local field F is Hausdorff, totally disconnected (Lemma 1.5), and locally compact (Corollary 3.5). We now study topological groups with these properties.

Definition 4.1. A *locally profinite* group is a topological group[1] which is Hausdorff, totally disconnected, and locally compact[2]. A compact locally profinite group is called *profinite*.

Example 4.2. (a) Discrete groups are locally profinite. Finite discrete groups are profinite.

(b) If G is (locally) profinite, then every closed subgroup is (locally) profinite, and every quotient of G by a closed normal subgroup is (locally) profinite.

(c) Arbitrary products of profinite groups are profinite. Finite products of locally profinite groups are locally profinite.

Exercise 4.3. Let G be a topological group and $H \subseteq G$ a closed subgroup.

(a) Assume G is compact. Show that H is open if and only if the index $[G : H]$ is finite.

(b) Show that H is open if and only if H contains a non-empty open subset of G.

(c) Show that every open subgroup of G is closed.

(d) Show that H is open in G if and only if the quotient topology on G/H is discrete.

Example 4.4. Let F be a local field and $n \in \mathbb{Z}_{\geqslant 1}$.

(a) F^n and $(F^\times)^n$ (endowed with the product topologies) are locally profinite groups with respect to addition and multiplication, respectively. The groups o_F, $\mathfrak{m}_F^n$, $o_F^\times$, and $(1+\mathfrak{m}_F^n)^\times$ are profinite.

(b) If R is a commutative unital ring, we denote by $\mathrm{Mat}_{n,n}(R) \cong R^{n^2}$ the ring of $n \times n$-matrices and by $\mathrm{GL}_n(R) \subseteq \mathrm{Mat}_{n,n}(R)$ the subset of invertible matrices.

For each $A \in \mathrm{Mat}_{n,n}(F)$, the determinant $\det(A) \in F$ is a polynomial in the entries of A. By Lemma 1.3(b), the map $\det\colon \mathrm{Mat}_{n,n}(F) \to F$ is continuous. Hence,

$$\mathrm{GL}_n(F) = \det^{-1}(F^\times)$$

is open in $\mathrm{Mat}_{n,n}(F)$. It follows that $\mathrm{GL}_n(F)$ is locally profinite.

The additive subgroup $\mathrm{Mat}_{n,n}(o_F) = o_F^{n^2}$ of $\mathrm{Mat}_{n,n}(F)$ is profinite.

Note that $\mathrm{GL}_n(o_F) = \det^{-1}(o_F^\times) \cap \mathrm{Mat}_{n,n}(o_F)$ is closed in $\mathrm{Mat}_{n,n}(o_F)$ hence compact, because $o_F^\times \subseteq o_F$ is closed by Lemma 1.5(a) and det is continuous. Thus, the open subgroup $\mathrm{GL}_n(o_F) \subseteq \mathrm{GL}_n(F)$ is profinite. For each $r \in \mathbb{Z}_{\geqslant 1}$, the *$r$-th congruence subgroup*

$$\begin{aligned} K_r &:= \mathrm{Ker}\big(\mathrm{GL}_n(o_F) \longrightarrow \mathrm{GL}_n(o_F/\mathfrak{m}_F^r)\big) \\ &= 1 + \varpi^r \,\mathrm{Mat}_{n,n}(o_F), \end{aligned}$$

is an open normal subgroup of $\mathrm{GL}_n(o_F)$, hence profinite. (For the equality, use that for each $A \in \mathrm{Mat}_{n,n}(o_F)$ we have $\det(1+\varpi^r A) \equiv 1 \mod \mathfrak{m}_F^r$, so that $\det(1+\varpi^r A) \in o_F^\times$ and hence $1+\varpi^r \,\mathrm{Mat}_{n,n}(o_F) \subseteq \mathrm{GL}_n(o_F)$.) The groups K_r, $r \in \mathbb{Z}_{\geqslant 1}$, form a fundamental system of open neighborhoods of 1.

[1]Recall that a *topological group* is a group G carrying a topology such that the map $G \times G \to G$, $(g,h) \mapsto gh^{-1}$ is continuous.

[2]A topological space X is called *locally compact* if for every $x \in X$ and every open neighborhood U of x there exists an open neighborhood V of x whose closure $\overline{V}$ is compact and contained in U.

Remark (Vedenissov's Theorem). If X is a totally disconnected, locally compact, Hausdorff topological space, then every point $x \in X$ admits a fundamental system of neighborhoods which are *clopen* (*i.e.*, open and closed).

Proof. Let $x \in X$ and let $U \ni x$ be an open neighborhood such that $\overline{U}$ is compact.

Step 1: Let $F \subseteq \overline{U}$ be a closed subset such that for every $y \in F$ there exists a clopen subset $C \subseteq \overline{U}$ with $y \in C$ and $x \notin C$. We claim there exists a clopen subset $C \subseteq \overline{U}$ with $F \subseteq C$ and $x \notin C$. Indeed, for each $y \in F$ let $C_y \subseteq \overline{U}$ be a clopen subset with $y \in C_y$ and $x \notin C_y$. Note that F is compact as a closed subset of the compact set $\overline{U}$. Hence, there exist $y_1, \ldots, y_r \in F$ with $F \subseteq \bigcup_{i=1}^r C_{y_i} =: C$, and $x \notin C$.

Step 2: Let $M = \bigcap_C C$, where C runs through the clopen subsets of $\overline{U}$ containing x. We first claim $M = \{x\}$; as X is totally disconnected, it suffices to prove that M is connected. Note that M is closed in $\overline{U}$ and $x \in M$. Consider closed (hence compact) subsets $E, F \subseteq \overline{U}$ with $M = E \cup F$ and $E \cap F = \varnothing$. Exchanging E and F if necessary, we may assume $x \in E$. We will show $F = \varnothing$. As X is Hausdorff, we find (by a standard argument) an open subset $W \subseteq X$ such that $E \subseteq W$ and $\overline{W} \cap F = \varnothing$. By construction, we have $\partial W \cap M = \varnothing$, where $\partial W := \overline{W} \smallsetminus W$ is the boundary of W. By the definition of M, this means that every point of $\partial W \cap \overline{U}$ can be separated from x by a clopen subset of $\overline{U}$. Step 1 provides a clopen subset $C \subseteq \overline{U}$ such that $\partial W \cap \overline{U} \subseteq C$ and $x \notin C$. By construction, we have $W \cap \overline{U} \smallsetminus C = \overline{W} \cap \overline{U} \smallsetminus C$, which is clopen in $\overline{U}$, contains x, and is disjoint from F. From $M \subseteq W \cap \overline{U} \smallsetminus C$ we deduce $M \cap F = \varnothing$, hence $F = \varnothing$. Therefore, M is connected, which proves $M = \{x\}$.

Note that $M = \{x\}$ is disjoint from $\partial U := \overline{U} \smallsetminus U$. Step 1 applied to ∂U yields a clopen subset $C \subseteq \overline{U}$ with $x \in C$ and $C \cap \partial U = \varnothing$. We finish by observing that C is clopen in X. ☐

Proposition 4.5. *For a topological group G, the following are equivalent:*

(i) G is profinite, i.e., compact, Hausdorff, and totally disconnected.

(ii) G is compact, Hausdorff, and the neutral element $1 \in G$ admits a fundamental system of open neighborhoods consisting of open normal subgroups.

(iii) For each open normal subgroup $N \subseteq G$ the quotient group G/N is finite, and the canonical map

$$G \longrightarrow \varprojlim_{N \subseteq G} G/N$$

is a topological group isomorphism, where N runs through a fundamental system of open neighborhoods of 1 consisting of normal subgroups of G.

Proof. "(i) $\Longrightarrow$ (ii)": Let $U \subseteq G$ be an open and closed neighborhood of 1. We have to construct an open normal subgroup $N \subseteq G$ such that $N \subseteq U$. Put $V := \{g \in U \mid Ug \subseteq U\}$. We first show that V is open. Fix $v \in V$ so that $Uv \subseteq U$. As multiplication is continuous, there exist for each $u \in U$ open neighborhoods U_u of u and V_u of v such that $U_u V_u \subseteq U$. Then $U = \bigcup_u U_u$ is an open covering. As U is compact as a closed subset of a compact set, we find $u_1, \ldots, u_r$ in U with $U = \bigcup_{i=1}^r U_{u_i}$. Then $W := \bigcap_{i=1}^r V_{u_i}$ is an open neighborhood of v and is contained in V, because it satisfies $U \cdot W = \bigcup_{i=1}^r U_{u_i} \cdot W \subseteq U$. Hence, V is open. Now put $H := V \cap V^{-1}$, which is also open. We have $1 \in H$. For all $g, h \in H$ we compute $Ugh^{-1} \subseteq Uh^{-1} \subseteq U$; this shows $gh^{-1} \in H$. Hence, H is an open subgroup of G which is contained in U. We find $g_1, \ldots, g_n \in G$ with $G = \bigcup_{i=1}^n g_i H$. Then $N := \bigcap_{i=1}^n g_i H g_i^{-1}$ is an open normal subgroup of G which is contained in U.

"(ii) $\Longrightarrow$ (iii)": Let $\mathcal{N}$ be a fundamental system of open neighborhoods of 1 consisting of normal subgroups of G viewed as a partially ordered set with respect to inclusion. We endow $\prod_{N\in\mathcal{N}} G/N$ with the product topology, where each G/N is discrete and finite. The topological group $\prod_N G/N$ is Hausdorff and compact (by Tychonoff's Theorem), and

$$\varprojlim_{N\in\mathcal{N}} G/N := \left\{ (g_N)_N \in \prod_{N\in\mathcal{N}} G/N \;\middle|\; \varphi_{N,N'}(g_{N'}) = g_N \text{ for all } N' \subseteq N \text{ in } \mathcal{N} \right\}$$

is a subgroup, where $\varphi_{N,N'}\colon G/N' \twoheadrightarrow G/N$ denotes the canonical projection for any $N' \subseteq N$ in $\mathcal{N}$. If $(g_N)_N \notin \varprojlim_N G/N$, there exist $N_1 \subseteq N_2$ in $\mathcal{N}$ with $\varphi_{N_2,N_1}(g_{N_1}) \neq g_{N_2}$. The open subset $\{g_{N_1}\} \times \{g_{N_2}\} \times \prod_{N\neq N_1,N_2} G/N$ does not intersect $\varprojlim_N G/N$. Hence, $\varprojlim_N G/N$ is closed in $\prod_N G/N$. The canonical map

$$\begin{aligned} \varphi\colon G &\longrightarrow \varprojlim_{N\in\mathcal{N}} G/N, \\ g &\longmapsto (gN)_N \end{aligned}$$

is well-defined and continuous. Since $\operatorname{Ker}(\varphi) = \bigcap_{N\in\mathcal{N}} N = \{1\}$, the map φ is injective. To prove surjectivity, let $(g_N N)_N \in \varprojlim_N G/N$ be arbitrary. We have to show

$$\bigcap_{N\in\mathcal{N}} g_N N \neq \varnothing, \tag{1.5}$$

because then any $g \in \bigcap_N g_N N$ satisfies $\varphi(g) = (g_N N)_N$. For all $N_1, \ldots, N_r \in \mathcal{N}$, there exists $N' \in \mathcal{N}$ with $N' \subseteq \bigcap_{i=1}^r N_i$, by assumption (ii). Then $g_{N'} N_i = g_{N_i} N_i$, for all $1 \leqslant i \leqslant r$, and therefore $g_{N'} \in \bigcap_{i=1}^r g_{N_i} N_i$ is non-empty. As each coset $g_N N$ is closed in G (the complement is open) and G is compact, we deduce (1.5). Since φ is continuous and bijective, G is compact, and $\varprojlim_N G/N$ is Hausdorff, it follows that φ is a homeomorphism.

"(iii) $\Longrightarrow$ (i)": Since each G/N is compact, Hausdorff, and totally disconnected, also the product $\prod_{N\in\mathcal{N}} G/N$ is compact (by Tychonoff's Theorem), Hausdorff, and totally disconnected. These properties are inherited by the closed subset $\varprojlim_N G/N$. □

Example 4.6. Let F be a local field and $\varpi \in o_F$ a uniformizer.

(a) The group $(o_F, +)$ is profinite, and $\{\mathfrak{m}_F^n\}_{n\geqslant 0}$ is a fundamental system of open neighborhoods of 0. Proposition 4.5 shows that the ring homomorphism

$$\begin{aligned} o_F &\xrightarrow{\cong} \varprojlim_{n\geqslant 0} o_F/\mathfrak{m}_F^n, \\ x &\longmapsto \left(x + \mathfrak{m}_F^n\right)_n \end{aligned}$$

is a homeomorphism. By virtue of Proposition 3.3, the map is given by $\sum_{i=0}^{\infty} a_i \varpi^i \mapsto \left(\sum_{i=0}^{n-1} a_i \varpi^i + \mathfrak{m}_F^n\right)_n$, which gives another proof of bijectivity.

As a special case, we find $\mathbb{Z}_p \cong \varprojlim_{n\geqslant 0} \mathbb{Z}/p^n\mathbb{Z}$, which gives another definition of $\mathbb{Z}_p$.

(b) Note that the map $o_F^\times \to (o_F/\mathfrak{m}_F^n)^\times$ is surjective with kernel $U_F^{(n)} := 1 + \mathfrak{m}_F^n$. Hence, from (a) we obtain topological group isomorphisms

$$o_F^\times \cong \big(\varprojlim_n o_F/\mathfrak{m}_F^n\big)^\times = \varprojlim_n (o_F/\mathfrak{m}_F^n)^\times \cong \varprojlim_n o_F^\times/U_F^{(n)}.$$

Exercise 4.7. Let G be a topological group. The following are equivalent:

(i) G is locally profinite.

(ii) G is Hausdorff and every open neighborhood of $1 \in G$ contains a compact open subgroup.

(iii) G contains an open subgroup which is profinite.

Exercise 4.8. Let G be a locally profinite group and $H \subseteq G$ a compact subgroup. Show that there exists a compact open subgroup $K \subseteq G$ containing H. (Hint: Let $K' \subseteq G$ be any compact open subgroup. Show that $K'' := \bigcap_{h \in H} hK'h^{-1}$ is still open and that $K := K''H$ is a compact open subgroup of G containing H.)

Example 4.9. Let L/F be an algebraic field extension. Then L/F is called *Galois* if every irreducible polynomial in $F[x]$ which has a root in L splits into *pairwise distinct* linear factors in $L[x]$.

We write $\mathcal{F}(L/F)$ for the set of intermediate fields of L/F which are finite Galois over F. Then

$$L/F \text{ is Galois} \iff L = \bigcup_{E \in \mathcal{F}(L/F)} E.$$

Let L/F be Galois. We denote $\mathrm{Gal}(L/F) := \mathrm{Aut}_F(L)$ the *Galois group* of L/F. The canonical map

$$\begin{aligned} \mathrm{Gal}(L/F) &\xrightarrow{\cong} \varprojlim_{E \in \mathcal{F}(L/F)} \mathrm{Gal}(E/F), \\ \sigma &\longmapsto \big(\sigma_{|E}\big)_E \end{aligned}$$

is an isomorphism of groups: The map is injective, because each $a \in L$ is contained in a finite Galois extension E/F. Given $(\sigma_E)_E \in \varprojlim_E \mathrm{Gal}(E/F)$, the σ_E's glue to a unique map $\sigma\colon L \to L$. It is clear that σ fixes F pointwise and is invertible, hence is an element of $\mathrm{Gal}(L/F)$.

We conclude from Proposition 4.5 that $\mathrm{Gal}(L/F)$ is a profinite group. The groups $\mathrm{Gal}(L/E)$, where E/F runs through the finite Galois extensions contained in L, are a fundamental system of open normal subgroups.

Exercise (Fundamental Theorem of Galois Theory). Let L/F be a Galois extension.

(a) L/E is Galois, for every intermediate field E of L/F.

(b) The maps

$$\begin{aligned} \{\text{closed subgroups of } \mathrm{Gal}(L/F)\} &\rightleftarrows \{\text{intermediate fields of } L/F\}, \\ H &\longmapsto L^H := \{a \in L \mid \sigma(a) = a \text{ for all } \sigma \in H\}, \\ \mathrm{Gal}(L/E) &\longleftarrow\!\shortmid E \end{aligned}$$

are bijective and inverse to each other.

(c) A subgroup $H \subseteq \mathrm{Gal}(L/F)$ is open if and only if L^H/F is finite.

(d) If E is an intermediate field of L/F, then E/F is Galois if and only if $\mathrm{Gal}(L/E)$ is a (closed) normal subgroup in $\mathrm{Gal}(L/F)$. In this case,

$$\begin{aligned} \mathrm{Gal}(L/F)/\,\mathrm{Gal}(L/E) &\xrightarrow{\cong} \mathrm{Gal}(E/F), \\ \sigma\,\mathrm{Gal}(L/E) &\longmapsto \sigma_{|E} \end{aligned}$$

is an isomorphism of topological groups.

Chapter 2

Smooth Representations of Locally Profinite Groups

§5. First Definitions and Examples

Let G be a locally profinite group. Denote $\mathbb{C}$ the field of complex numbers.

Definition 5.1. (a) A *G-representation* is a pair (V, π) consisting of a $\mathbb{C}$-vector space V together with a group homomorphism

$$\pi\colon G \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V).$$

We sometimes write V or π instead of (V, π) and $gv := \pi(g)v$, for $g \in G$, $v \in V$.

Equivalently, a G-representation is $\mathbb{C}$-vector space V together with a map $\Phi\colon G \times V \to V$, $(g, v) \mapsto g \cdot v$ such that $1 \cdot v = v$, $(gh) \cdot v = g \cdot (h \cdot v)$ and $\Phi(g, \_)\colon V \to V$ is $\mathbb{C}$-linear for all $v \in V$, $g, h \in G$.

Given G-representations (V, π) and (W, ρ), a $\mathbb{C}$-linear map $f\colon V \to W$ is called *G-equivariant* if $f(gv) = gf(v)$, for all $v \in V$, $g \in G$. We denote $\mathrm{Hom}_G(V, W)$ the $\mathbb{C}$-vector space of all G-equivariant $\mathbb{C}$-linear maps.

(b) A G-representation (V, π) is called *smooth* if for all $v \in V$ the *stabilizer*

$$\mathrm{Stab}_G(v) := \{g \in G \mid gv = v\}$$

is an open subgroup of G.

We denote

$$\mathrm{Rep}(G)$$

the category of smooth G-representations together with G-equivariant maps.

Lemma 5.2. *Let (V, π) be a G-representation. The following conditions are equivalent:*

(i) (V, π) *is smooth.*

(ii) $V = \bigcup_{K \subseteq G} V^K$, *where* $V^K := \{v \in V \mid gv = v \text{ for all } g \in K\}$ *and* K *runs through all compact open subgroups of* G.

(iii) The action map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous, when V is endowed with the discrete topology and $G \times V$ with the product topology.

Proof. "(i) $\Longrightarrow$ (iii)": Let $(g, v) \in G \times V$. Then $g \operatorname{Stab}_G(v) \times \{v\}$ is an open neighborhood of (g, v) such that $\pi(g \operatorname{Stab}_G(v))(\{v\}) = \{gv\}$. Hence, the action map is continuous.

"(iii) $\Longrightarrow$ (ii)": Let $v \in V$ and denote the action map by Φ. As $\Phi^{-1}(\{v\}) \subseteq G \times V$ is open, there exists (by Exercise 4.7) an open compact subgroup K of G such that $\Phi(K \times \{v\}) \subseteq \{v\}$. In other words, $v \in V^K$.

"(ii) $\Longrightarrow$ (i)": Let $v \in V$. By assumption, there exists a compact open subgroup K of G with $K \subseteq \operatorname{Stab}_G(v)$, and $\operatorname{Stab}_G(v) = \bigcup_{g \in \operatorname{Stab}_G(v)} gK$ is open.

□

Example 5.3. (a) A group homomorphism $\chi\colon G \to \mathbb{C}^\times$ is called a *character*. Then $(\mathbb{C}, \chi)$ is smooth if and only if $\operatorname{Ker}(\chi)$ is open. The *trivial representation* is the G-representation $(\mathbb{C}, \mathbf{1})$, where $\mathbf{1}(g) = 1$ for all $g \in G$.

(b) Let $G = \mathrm{GL}_1(F) = F^\times$ for a local field F with uniformizer ϖ. Since $F^\times = \varpi^{\mathbb{Z}} \times o_F^\times$ (Lemma 3.2), giving a smooth character $\chi\colon F^\times \to \mathbb{C}^\times$ is the same as giving:

- a complex number $\chi(\varpi) \in \mathbb{C}^\times$;
- a character $o_F^\times/(1 + \mathfrak{m}_F^r)^\times \to \mathbb{C}^\times$.

(c) Let $C_c^\infty(G)$ be the $\mathbb{C}$-vector space of all functions $f\colon G \to \mathbb{C}$ which are locally constant and have compact *support*

$$\operatorname{Supp}(f) \coloneqq \overline{\{g \in G \mid f(g) \neq 0\}},$$

where the overline means topological closure. The $\mathbb{C}$-vector space structure is given pointwise, that is, $(f_1 + f_2)(g) \coloneqq f_1(g) + f_2(g)$ and $(af)(g) \coloneqq a \cdot f(g)$, for all $f, f_1, f_2 \in C_c^\infty(G)$, $a \in \mathbb{C}$, and $g \in G$. The group G acts on $C_c^\infty(G)$ by *right translation*:

$$\big(\rho(g)f\big)(g') \coloneqq f(g'g), \qquad \text{for all } f \in C_c^\infty(G),\ g, g' \in G.$$

We claim that $(C_c^\infty(G), \rho)$ is a smooth G-representation. For each compact open subgroup K, we put

$$C_c^\infty(G/K) \coloneqq C_c^\infty(G)^K;$$

these are precisely the functions $f \in C_c^\infty(G)$ which satisfy $f(gk) = f(g)$ for all $g \in G$, $k \in K$. Let $f \in C_c^\infty(G)$. For each $g \in \operatorname{Supp}(f)$ there exists a compact open subgroup $K_g \subseteq G$ such that f is constant on gK_g; in particular $gK_g \subseteq \operatorname{Supp}(f)$. As $\operatorname{Supp}(f)$ is compact, we find $g_1, \dots, g_r \in \operatorname{Supp}(f)$ with $\operatorname{Supp}(f) = \bigcup_{i=1}^r g_i K_{g_i}$.[1] For $K \coloneqq \bigcap_{i=1}^r K_{g_i}$ we have $f \in C_c^\infty(G/K)$. In other words,

$$C_c^\infty(G) = \bigcup_K C_c^\infty(G/K), \tag{2.1}$$

where $K \subseteq G$ runs through the compact open subgroups.

Analogously, G acts on $C_c^\infty(G)$ by *left translation*,

$$\big(\lambda(g)f\big)(g') \coloneqq f(g^{-1}g'), \qquad \text{for all } g, g' \in G,$$

and a similar argument as above shows that $(C_c^\infty(G), \lambda)$ is smooth. Note that each $C_c^\infty(G/K)$ is a (smooth) subrepresentation of $(C_c^\infty(G), \lambda)$.

[1] This argument shows that $\operatorname{Supp}(f)$ is open (and closed).

(d) If (V, π) is a smooth G-representation and $W \subseteq V$ is a G-invariant subspace, then W and V/W are smooth G-representations.

(e) If $\{(V_i, \pi_i)\}_{i \in I}$ is a family in $\operatorname{Rep}(G)$, then the direct sum $\bigoplus_{i \in I} V_i$ is a smooth G-representation.

(f) If (V, π) and (W, σ) are smooth G-representations, then $(\pi \otimes \sigma)(g)(v \otimes w) := \pi(g)v \otimes \sigma(g)w$ defines on $V \otimes_{\mathbb{C}} W$ the structure of a smooth G-representation.

(g) Let $H \subseteq G$ be a closed subgroup. If (V, π) is a smooth representation of G, then $(V, \pi_{|H})$ is a smooth representation of H called the *restriction* of (V, π).

Exercise 5.4. (a) Let (V, π) be a (not necessarily smooth) G-representation and put

$$V^\infty := \bigcup_{K \subseteq G} V^K,$$

where K runs through the compact open subgroups of G. Show that (V^∞, π) is the largest smooth subrepresentation of V (in particular G-invariant and a $\mathbb{C}$-subvector space).

(b) Let $f\colon (V, \pi) \to (W, \sigma)$ be a G-equivariant homomorphism between G-representations. Show that $f(V^\infty) \subseteq W^\infty$. Deduce that the assignment $V \mapsto V^\infty$ is functorial.

Exercise 5.5. Find a locally profinite group G and a family $\{(V_i, \pi_i)\}_{i \in I}$ in $\operatorname{Rep}(G)$ such that the cartesian product $\prod_{i \in I} V_i$ is not smooth.
(Hint: Consider the $\mathbb{Z}_p$-representation $\prod_{n \in \mathbb{Z}_{\geqslant 1}} C_c^\infty(\mathbb{Z}_p/p^n\mathbb{Z}_p)$.)

Definition 5.6. A G-representation (V, π) is called *irreducible* if V has precisely two subrepresentations, namely $\{0\}$ and V.[2] We denote $\mathbf{Irr}(G)$ the set of isomorphism classes of irreducible smooth G-representations.

Lemma 5.7. *Assume G is profinite. If (V, π) is a smooth irreducible G-representation, then V is finite dimensional.*

Proof. Fix $v \in V$, $v \neq 0$. There exists an open normal subgroup $K \subseteq G$ with $v \in V^K$. Then $[G : K]$ is finite and hence the subspace $W := \sum_{gK \in G/K} \mathbb{C}\pi(g)v \subseteq V$ is (well-defined and) G-invariant. As V is irreducible, we conclude that $V = W$, which has dimension $\leqslant [G : K]$. □

Remark. The proof of the lemma shows that the irreducible smooth representations of a profinite group G are given by the irreducible representations of G/K, where K runs through the open normal subgroups of G. In this way, the representation theory of finite groups enters the smooth representation theory of profinite groups.

Lemma 5.8. *Let $K \subseteq G$ be a compact subgroup. The functor* $\operatorname{Rep}(G) \to \operatorname{Vect}_{\mathbb{C}}$, $V \mapsto V^K$ *is* exact*: Let $V' \xrightarrow{\phi} V \xrightarrow{\psi} V''$ be an exact sequence of G-equivariant homomorphisms, which means that* $\operatorname{Im}(\varphi) = \operatorname{Ker}(\psi)$. *Then the induced sequence*

$$(V')^K \xrightarrow{\phi^K} V^K \xrightarrow{\psi^K} (V'')^K$$

is exact.

[2] This also means that $\{0\}$ is not irreducible.

Proof. Since $\psi \circ \phi = 0$, it is clear that $\operatorname{Im}(\phi^K) \subseteq \operatorname{Ker}(\psi^K)$. Conversely, let $v \in V^K$ with $\psi(v) = 0$. Since $\operatorname{Im}(\varphi) = \operatorname{Ker}(\psi)$, there exists $v' \in V'$ with $\varphi(v') = v$. As V' is smooth, we find an open subgroup $H' \subseteq G$ with $v' \in (V')^{H'}$. Put $H \coloneqq K \cap H'$ so that also $v' \in (V')^H$. Then $v_0' \coloneqq \frac{1}{[K:H]} \sum_{k \in K/H} kv'$ lies in $(V')^K$, and

$$\phi^K(v_0') = \frac{1}{[K:H]} \sum_{k \in K/H} \phi(kv') = \frac{1}{[K:H]} \sum_{k \in K/H} k\phi(v') = \frac{1}{[K:H]} \sum_{k \in K/H} v = v,$$

where "$k \in K/H$" means that k runs through a set of representatives of K/H, and that the sum is finite and independent of this choice. Hence, $\operatorname{Im}(\varphi^K) = \operatorname{Ker}(\psi^K)$. □

Exercise 5.9. Show that a sequence $V' \to V \to V''$ in $\operatorname{Rep}(G)$ is exact if and only if the induced sequence $(V')^K \to V^K \to (V'')^K$ is exact for all compact open subgroups K of G.

§6. Haar Measures

Let G be a locally profinite group.

Exercise 6.1. The *group algebra* $\mathbb{C}[G]$ is defined as the $\mathbb{C}$-vector space on the basis $\{e_g\}_{g \in G}$ and with multiplication given by bilinear extension of the multiplication on G:

$$\Big(\sum_{g \in G} a_g e_g\Big) \cdot \Big(\sum_{g \in G} b_g e_g\Big) \coloneqq \sum_{g,h \in G} a_g b_h \cdot e_{gh} = \sum_{g \in G} \Big(\sum_{h \in G} a_h b_{h^{-1}g}\Big) \cdot e_g.$$

(a) Show that $\mathbb{C}[G]$ is a unital, associative $\mathbb{C}$-algebra satisfying $G \subseteq \mathbb{C}[G]^\times$.

(b) Let V be a $\mathbb{C}$-vector space. Show that giving a group homomorphism $G \to \operatorname{Aut}_{\mathbb{C}}(V)$ is equivalent to giving a unital $\mathbb{C}$-algebra homomorphism $\mathbb{C}[G] \to \operatorname{End}_{\mathbb{C}}(V)$, where $\operatorname{End}_{\mathbb{C}}(V)$ denotes the $\mathbb{C}$-algebra of $\mathbb{C}$-linear endomorphisms on V with respect to composition.

 In other words, a G-representation is the same as a $\mathbb{C}[G]$-module.

In view of the exercise, we ask whether we can identify *smooth* G-representations with modules over some $\mathbb{C}$-algebra. This is indeed the case, but the answer turns out to be much more involved than for abstract representations. This section serves as a preparation.

Definition 6.2. Recall the smooth G-representation $\big(C_c^\infty(G), \lambda\big)$ from Example 5.3. A *left Haar measure* is a non-zero $\mathbb{C}$-linear map $\mu_G \colon C_c^\infty(G) \to \mathbb{C}$ satisfying the following properties:

(i) $\mu_G\big(\lambda(g)f\big) = \mu_G(f)$ for all $g \in G$, $f \in C_c^\infty(G)$;

(ii) $\mu_G(f) \geqslant 0$ for all $f \in C_c^\infty(G)$ with $\operatorname{Im}(f) \subseteq \mathbb{R}_{\geqslant 0}$.

A *right Haar measure* is defined analogously.

Notation. For each compact open subset $X \subseteq G$ we denote $\mathbf{1}_X \in C_c^\infty(G)$ the *characteristic function* of X.

Lemma 6.3. *The G-representation* $(C_c^\infty(G), \lambda)$ *is generated by the* $\mathbf{1}_K$*, where* $K \subseteq G$ *runs through the compact open subgroups. Explicitly, for each* $f \in C_c^\infty(G/K)$ *one has*

$$f = \sum_{g \in G/K} f(g) \cdot \lambda(g)\mathbf{1}_K,$$

where "$g \in G/K$*" means that g runs through a set of representatives of* G/K*, and that the sum is finite and independent of this choice.*

Proof. Obvious. □

Proposition 6.4. *Up to multiplication by a constant* $c > 0$*, there exists a unique left (resp. right) Haar measure* $\mu_G \colon C_c^\infty(G) \to \mathbb{C}$*.*

Proof. Let $\mu_G \colon C_c^\infty(G) \to \mathbb{C}$ be a left Haar measure. Fix a compact open subgroup $K \subseteq G$ so that $\mu_G(\mathbf{1}_K) \in \mathbb{R}_{>0}$. We claim that $\mu_G(\mathbf{1}_K)$ uniquely determines μ_G. If $H \subseteq K$ is any open subgroup, then $\mathbf{1}_K = \sum_{k \in K/H} \mathbf{1}_{kH} = \sum_{k \in K/H} \lambda(k)\mathbf{1}_H$. We compute

$$\mu_G(\mathbf{1}_K) = \sum_{k \in K/H} \mu_G\big(\lambda(k)\mathbf{1}_H\big) = \sum_{k \in K/H} \mu_G(\mathbf{1}_H) = [K : H] \cdot \mu_G(\mathbf{1}_H). \tag{2.2}$$

Now, let $f \in C_c^\infty(G)$ be arbitrary. There exists a compact open subgroup $H \subseteq K$ with $f \in C_c^\infty(G/H)$. Write $f = \sum_{g \in G/H} f(g) \cdot \lambda(g)\mathbf{1}_H$ as in Lemma 6.3; we deduce from (2.2) that

$$\mu_G(f) = \sum_{g \in G/H} f(g) \cdot \mu_G(\mathbf{1}_H) = \frac{1}{[K : H]} \sum_{g \in G/H} f(g) \cdot \mu_G(\mathbf{1}_K). \tag{2.3}$$

This shows that μ_G is unique up to multiplication by a positive scalar.

For the existence, we fix a compact open subgroup $K \subseteq G$ and choose $\mu_G(\mathbf{1}_K) \in \mathbb{R}_{>0}$. If $f \in C_c^\infty(G)$ is any element, we write $f = \sum_{g \in G/H} f(g)\mathbf{1}_{gH}$ for some compact open subgroup $H \subseteq G$ with $f \in C_c^\infty(G/H)$ and define $\mu_G(f)$ as in (2.3). It remains to see that $\mu_G(f)$ is independent of the choice of H. Let $U \subseteq K$ be another subgroup with $f \in C_c^\infty(G/U)$. By replacing U with $U \cap H$ if necessary, we may assume $U \subseteq H$. Write $\mathbf{1}_H = \sum_{h \in H/U} \mathbf{1}_{hU}$. Check that, if g and h run through a system of representatives for G/H and H/U, respectively, then gh runs through a system of representatives for G/U. Then $f = \sum_{g \in G/H} \sum_{h \in H/U} f(gh)\mathbf{1}_{ghU}$ and

$$\begin{aligned}\frac{1}{[K : U]} \sum_{g \in G/H} \sum_{h \in H/U} f(gh)\mu_G(\mathbf{1}_K) &= \frac{1}{[K : H] \cdot [H : U]} \sum_{g \in G/H} \sum_{h \in H/U} f(g)\mu_G(\mathbf{1}_K) \\ &= \frac{1}{[K : H]} \sum_{g \in G/H} f(g)\mu_G(\mathbf{1}_K).\end{aligned}$$

Hence, $\mu_G(f)$ is well-defined. The properties (i) and (ii) for μ_G are obvious from (2.3). □

Notation. Let μ_G be a left Haar measure. For each $f \in C_c^\infty(G)$ we write

$$\int_G f(x)\, \mathrm{d}\mu_G(x) \coloneqq \mu_G(f).$$

The invariance under left translation then reads $\int_G f(gx)\,\mathrm{d}\mu_G(x) = \int_G f(x)\,\mathrm{d}\mu_G(x)$ or, more informally, $\mathrm{d}\mu_G(x) = \mathrm{d}\mu_G(gx)$ for all $g \in G$.

If $X \subseteq G$ is a compact open subset, we call

$$\operatorname{vol}(X) \coloneqq \operatorname{vol}(X; \mu_G) \coloneqq \mu_G(\mathbf{1}_X)$$

the *volume* of X with respect to μ_G.

Exercise 6.5. Let μ_G be a left Haar measure.

(a) For any two compact open subgroups $H, K \subseteq G$ we have

$$\frac{\operatorname{vol}(K)}{\operatorname{vol}(H)} = [K : H] \coloneqq \frac{[K : K \cap H]}{[H : K \cap H]},$$

called the *generalized index* of H in K.

(b) Let $g \in G$. Show that the function $\nu_G\colon C_c^\infty(G) \to \mathbb{C}$, $f \mapsto \mu_G\big(\rho(g)f\big)$ defines a left Haar measure. Hence, there exists a unique $\delta_G(g) \in \mathbb{R}_{>0}$ with $\nu_G = \delta_G(g)\mu_G$. In integral notation:

$$\int_G f(xg)\,\mathrm{d}\mu_G(x) = \delta_G(g) \int_G f(x)\,\mathrm{d}\mu_G(x).$$

More informally, we have $\mathrm{d}\mu_G(x) = \delta_G(g)\mu_G(xg)$ for all $g \in G$.

(c) Show $\delta_G(gh) = \delta_G(g)\delta_G(h)$ for all $g, h \in G$. Hence, $\delta_G\colon G \to \mathbb{R}_{>0}^\times$ is a character, called the *modulus character*.

(d) Let $K \subseteq G$ be any compact open subgroup. Show that $\delta_G(g) = [gKg^{-1} : K] \in \mathbb{Q}_{>0}^\times$ for all $g \in G$. In particular, δ_G is trivial on every compact subgroup and hence defines a smooth character $\delta_G\colon G \to \mathbb{C}^\times$ which is independent of μ_G. (See also Exercise 4.8.)

(e) Show that $\nu_G(f) \coloneqq \mu_G(\delta_G \cdot f)$ defines a right Haar measure ν_G on G. (Here, we define $(\delta_G \cdot f)(g) = \delta_G(g) \cdot f(g)$ for all $f \in C_c^\infty(G)$ and $g \in G$.)

(f) Let H be another locally profinite group. Show that $\delta_{G\times H}\big((g,h)\big) = \delta_G(g) \cdot \delta_H(h)$ for all $(g,h) \in G \times H$.

Exercise. Let $H \subseteq G$ be a closed subgroup and let $\theta\colon H \to \mathbb{C}^\times$ be a smooth character. Let $C_c^\infty(H\backslash G, \theta)$ be the space of locally constant functions $f\colon G \to \mathbb{C}$ with compact support in the coset space $H\backslash G$ which satisfy $f(hg) = \theta(h)f(g)$ for all $h \in H$, $g \in G$. Note that $C_c^\infty(H\backslash G, \theta)$ becomes a smooth G-representation if we let G act via right translation.

We fix a left Haar measure μ_H on H and a right Haar measure ν_G on G.

(a) Show that the map

$$\begin{aligned} \Theta\colon \big(C_c^\infty(G), \rho\big) &\longrightarrow \big(C_c^\infty(H\backslash G, \theta), \rho\big), \\ f &\longmapsto \Big[g \mapsto \int_H \delta_H(h)\theta(h^{-1})f(hg)\,\mathrm{d}\mu_H(h)\Big] \end{aligned}$$

is a surjective G-equivariant homomorphism. (Hint: For surjectivity, it suffices to prove that the induced map $C_c^\infty(G)^K \to C_c^\infty(H\backslash G, \theta)^K$ is surjective for all compact open subgroups $K \subseteq G$.)

(b) Show that $\Theta(\lambda(h)f) = \delta_H(h)\theta(h^{-1}) \cdot \Theta(f)$, for all $h \in H$ and $f \in C_c^\infty(H\backslash G, \theta)$.

(c) Show that the following are equivalent:

(i) $\nu_G \colon C_c^\infty(G) \to \mathbb{C}$ factors through a $\mathbb{C}$-linear map $\nu_{H\backslash G} \colon C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$ satisfying $\nu_{H\backslash G}(\rho(g)f) = \nu_{H\backslash G}(f)$ for all $g \in G$, $f \in C_c^\infty(H\backslash G, \theta)$;

(ii) $\theta = \delta_H \cdot (\delta_G^{-1})_{|H}$.

If these conditions are satisfied, $\nu_{H\backslash G} \colon C_c^\infty(H\backslash G, \theta) \to \mathbb{C}$ is called a *semi-invariant Haar measure* on $H\backslash G$; it is unique up to multiplication by a non-zero scalar. One writes

$$\nu_{H\backslash G}(f) =: \int_{H\backslash G} f(g)\,\mathrm{d}\nu_{H\backslash G}(g), \qquad \text{for all } f \in C_c^\infty(H\backslash G, \theta).$$

Fubini's Theorem 6.6. *Let G, H be locally profinite groups, and let μ_G, μ_H be left Haar measures on G, H, respectively. There exists a unique left Haar measure $\mu_G \otimes \mu_H \colon C_c^\infty(G \times H) \to \mathbb{C}$ such that*

$$(\mu_G \otimes \mu_H)(f \otimes f') = \mu_G(f) \cdot \mu_H(f'), \tag{2.4}$$

for all $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(H)$, where $(f \otimes f')(g,h) := f(g) \cdot f'(h)$. For all $\Phi \in C_c^\infty(G \times H)$ we have

$$\begin{aligned}\int_H \int_G \Phi(x,y)\,\mathrm{d}\mu_G(x)\,\mathrm{d}\mu_H(y) &= \int_{G\times H} \Phi(x,y)\,\mathrm{d}(\mu_G \otimes \mu_H)(x,y) \\ &= \int_G \int_H \Phi(x,y)\,\mathrm{d}\mu_H(y)\,\mathrm{d}\mu_G(x).\end{aligned} \tag{2.5}$$

Proof. Let $\Phi \in C_c^\infty(G \times H)$. There exist compact open subgroups $K \subseteq G$ and $U \subseteq H$ such that Φ factors through a function $G/K \times H/U \to \mathbb{C}$ with finite support. Hence, we have

$$\Phi = \sum_{g\in G/K} \sum_{h \in H/U} \Phi(g,h) \cdot \mathbf{1}_{gK} \otimes \mathbf{1}_{hU}.$$

We deduce that the map $C_c^\infty(G) \otimes_{\mathbb{C}} C_c^\infty(H) \xrightarrow{\cong} C_c^\infty(G \times H)$ given by $f \otimes f' \mapsto [(g,h) \mapsto f(g) \cdot f'(h)]$ is an isomorphism of $G \times H$-representations. Hence, the composite

$$C_c^\infty(G \times H) \xleftarrow{\cong} C_c^\infty(G) \otimes_{\mathbb{C}} C_c^\infty(H) \xrightarrow{\mu_G \otimes \mu_H} \mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$$

defines the unique left Haar measure on $G \times H$ satisfying (2.4). Property (2.5) can then be checked for $\Phi = f \otimes f'$ with $f \in C_c^\infty(G)$ and $f' \in C_c^\infty(H)$ in which case it is a restatement of

$$\mu_H\big(\mu_G(f) \cdot f'\big) = \mu_G(f) \cdot \mu_H(f') = \mu_G\big(\mu_H(f') \cdot f\big).$$

□

§7. The Hecke Algebra

Let G be a locally profinite group and fix a left Haar measure $\mu_G \colon C_c^\infty(G) \to \mathbb{C}$.

Definition 7.1. We define on the $\mathbb{C}$-vector space $\mathcal{H}(G) \coloneqq C_c^\infty(G)$ a *convolution product* as follows: Let $f, f' \in \mathcal{H}(G)$. The map $G \times G \to \mathbb{C}$, $(x,g) \mapsto f(x)f'(x^{-1}g)$ defines an element of $C_c^\infty(G \times G)$; for all $g \in G$ set

$$\begin{aligned}(f *_{\mu_G} f')(g) &\coloneqq \int_G f(x)f'(x^{-1}g)\,\mathrm{d}\mu_G(x)\\ &= \int_G f(gy)f'(y^{-1})\,\mathrm{d}\mu_G(y) \qquad \text{(substitute } x = gy\text{)}.\end{aligned}$$

Then $f * f' \coloneqq f *_{\mu_G} f'$ lies in $\mathcal{H}(G)$.

We call $\mathcal{H}(G)$ the *Hecke algebra* of G.

Exercise. (a) Use Fubini's Theorem to check that $(\mathcal{H}(G), *)$ is an (in general non-unital) associative $\mathbb{C}$-algebra.

(b) Let ν_G be another left Haar measure. Show that the $\mathbb{C}$-algebras $\big(\mathcal{H}(G), *_{\nu_G}\big)$ and $\big(\mathcal{H}(G), *_{\mu_G}\big)$ are isomorphic.

Example 7.2. If G is discrete, then $\mathcal{H}(G) \cong \mathbb{C}[G]$ as $\mathbb{C}$-algebras. In fact, $\mathcal{H}(G)$ has a unit if and only if G is discrete.

Lemma 7.3. *For every $g \in G$ and $f, f' \in \mathcal{H}(G)$ one has:*

(a) $\rho(g)(f * f') = f * \big(\rho(g)f'\big)$*;*

(b) $\lambda(g)(f * f') = \big(\lambda(g)f\big) * f'$*;*

(c) $\big(\rho(g)f\big) * f' = \delta_G(g) \cdot f * \big(\lambda(g^{-1})f'\big)$*, where δ_G is the modulus character from Exercise 6.5.*

Proof. (a) and (b) follow immediately from the first and second formula for the convolution product, respectively. For (c), we compute, for any $h \in G$:

$$\begin{aligned}\big(\big(\rho(g)f\big) * f'\big)(h) &= \int_G f(xg)f'(x^{-1}h)\,\mathrm{d}\mu_G(x) = \delta_G(g)\int_G f(xg)f'(x^{-1}h)\,\mathrm{d}\mu_G(xg)\\ &= \delta_G(g)\int_G f(y)f'(gy^{-1}h)\,\mathrm{d}\mu_G(y) = \delta_G(g)\cdot\big(f * \big(\lambda(g^{-1})f'\big)\big)(h).\end{aligned}$$ □

Proposition 7.4. *For each compact open subset $X \subseteq G$, put*

$$e_X \coloneqq \mathrm{vol}(X; \mu_G)^{-1} \cdot \mathbf{1}_X \in \mathcal{H}(G).$$

Let $K \subseteq G$ be a compact open subgroup.

(a) For each open subgroup $H \subseteq K$, one has $e_H * e_K = e_K = e_K * e_H$. In particular, e_K is an idempotent.

(b) A function $f \in \mathcal{H}(G)$ satisfies $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K$, $g \in G$. Similarly, $f * e_K = f$ if and only if $f(gk) = f(g)$ for all $k \in K$, $g \in G$.

(c) The space $\mathcal{H}(G,K) \coloneqq e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$ with unit e_K. It consists of all functions $f \in \mathcal{H}(G)$ with $f(kgk') = f(g)$ for all $k, k' \in K$, $g \in G$.

Proof. Let $g \in G$. Note that the function $x \mapsto \mathbf{1}_H(x)\mathbf{1}_K(x^{-1}g)$ is the characteristic function of $H \cap gK$. Hence,

$$\begin{aligned}(e_H * e_K)(g) &= \operatorname{vol}(H)^{-1}\operatorname{vol}(K)^{-1}\int_G \mathbf{1}_H(x)\mathbf{1}_K(x^{-1}g)\,\mathrm{d}\mu_G(x)\\ &= \frac{\operatorname{vol}(H\cap gK)}{\operatorname{vol}(H)\operatorname{vol}(K)} = e_K(g).\end{aligned}$$

A similar computation shows $e_K * e_H = e_K$, which proves (a). To prove (b), let $f \in \mathcal{H}(G)$. The function $e_K * f$ is left K-invariant by Lemma 7.3(b). This shows that, if $e_K * f = f$, then f is left K-invariant. Conversely, if f is left K-invariant, then for any $g \in G$ the function $x \mapsto \mathbf{1}_K(x)f(x^{-1}g)$ coincides with $f(g)\cdot\mathbf{1}_K$, and hence

$$\begin{aligned}(e_K * f)(g) &= \operatorname{vol}(K)^{-1}\int_G \mathbf{1}_K(x)f(x^{-1}g)\,\mathrm{d}\mu_G(x)\\ &= \operatorname{vol}(K)^{-1}\cdot\mu_G(\mathbf{1}_K)\cdot f(g) = f(g).\end{aligned}$$

The remaining assertions in (b) are analogous.

Finally, (c) follows at once from (a) and (b). □

Remark. It follows from Proposition 7.4(b) that for all $f_1,\dots,f_n \in \mathcal{H}(G)$ there exists an idempotent $e_K \in \mathcal{H}(G)$ with $e_K * f_i = f_i = f_i * e_K$ for all i. Even though $\mathcal{H}(G)$ does not admit a unit, it has many idempotents. Such $\mathbb{C}$-algebras are called *idempotented*.

Definition 7.5. (a) An $\mathcal{H}(G)$-*module* is a $\mathbb{C}$-vector space V together with a $\mathbb{C}$-linear map

$$\begin{aligned}\mathcal{H}(G)\otimes_{\mathbb{C}} V &\longrightarrow V,\\ f\otimes v &\longmapsto \pi(f)v\end{aligned}$$

which satisfies $\pi(f)(\pi(f')v) = \pi(f*f')v$, for all $f, f' \in \mathcal{H}(G)$ and $v \in V$. More concisely, an $\mathcal{H}(G)$-module is a (non-unital) $\mathbb{C}$-algebra homomorphism $\pi\colon \mathcal{H}(G) \to \operatorname{End}_{\mathbb{C}}(V)$. We often write $f * v$ for $\pi(f)v$.

A $\mathbb{C}$-linear map $\alpha\colon V \to V'$ between $\mathcal{H}(G)$-modules is called $\mathcal{H}(G)$-*linear* if $\alpha(f*v) = f*\alpha(v)$, for all $v \in V$, $f \in \mathcal{H}(G)$.

(b) An $\mathcal{H}(G)$-module V is called *smooth* if $\mathcal{H}(G)*V = V$, *i.e.*, for all $v \in V$ there exist $f_1,\dots,f_n \in \mathcal{H}(G)$ and $v_1,\dots,v_n \in V$ such that $v = \sum_{i=1}^n f_i * v_i$.

We denote

$$\operatorname{Mod}(\mathcal{H}(G))$$

the category with objects the smooth $\mathcal{H}(G)$-modules and morphisms the $\mathcal{H}(G)$-linear maps.

Exercise 7.6. Let V be an $\mathcal{H}(G)$-module. Show that the following assertions are equivalent:

(i) V is smooth.

(ii) For all $v \in V$ there exists a compact open subgroup $K \subseteq G$ such that $e_K * v = v$.

Deduce that $\mathcal{H}(G)$ is a smooth $\mathcal{H}(G)$-module.

Theorem 7.7. *There is an isomorphism of categories*

$$\mathrm{Rep}(G) \xrightarrow{\cong} \mathrm{Mod}(\mathcal{H}(G)),$$

which is the identity on objects and morphisms.

Proof. Step 1: Let $(V, \pi) \in \mathrm{Rep}(G)$ be a smooth representation. We construct on V the structure of a smooth $\mathcal{H}(G)$-module.

First, it is convenient to introduce some notation. Denote $C_c^\infty(G, V)$ the $\mathbb{C}$-vector space of functions $f : G \to V$ which are locally constant and have compact support. Then $C_c^\infty(G, V)$ becomes a smooth G-representation via $(gf)(g') := \pi(g)f(g^{-1}g')$, for all $g, g' \in G$, $f \in C_c^\infty(G, V)$.

We claim that the map

$$\begin{aligned} C_c^\infty(G) \otimes_{\mathbb{C}} V &\xrightarrow{\cong} C_c^\infty(G, V), \\ f \otimes v &\longmapsto [g \mapsto f(g)v] \end{aligned} \tag{2.6}$$

is an isomorphism of smooth G-representations, where G acts diagonally on the left hand side via $g \cdot (f \otimes v) = \lambda(g)f \otimes \pi(g)v$.

Let $\Phi = \sum_{i=1}^n f_i \otimes v_i \in C_c^\infty(G) \otimes_{\mathbb{C}} V$ be in the kernel of (2.6). Since V admits a $\mathbb{C}$-basis, we may assume that $v_1, \ldots, v_n$ are linearly independent. But then the condition that Φ is in the kernel is equivalent to $f_i = 0$ for all i. Hence $\Phi = 0$.

We show surjectivity. Let $f \in C_c^\infty(G, V)$ be arbitrary. Since f is locally constant and has compact support, we find a compact open subgroup $K \subseteq G$ such that $f(gk) = f(g)$, for all $g \in G$, $k \in K$. Then f is the image of $\sum_{g \in G/K} \mathbf{1}_{gK} \otimes f(g)$ (the sum is finite, because f has compact support).

Now, there exists a unique G-equivariant map $\mu_G : C_c^\infty(G, V) \to V$ making the diagram

$$\begin{array}{ccc} C_c^\infty(G) \otimes_{\mathbb{C}} V & \xrightarrow{\mu_G \otimes \mathrm{id}_V} & \mathbb{C} \otimes_{\mathbb{C}} V \\ {\scriptstyle\cong}\big\downarrow & & \big\| \\ C_c^\infty(G, V) & \xrightarrow[\mu_G]{} & V \end{array}$$

commute. For each $f \in C_c^\infty(G, V)$, we write

$$\int_G f(x)\, \mathrm{d}\mu_G(x) := \mu_G(f) \in V.$$

We now define the $\mathcal{H}(G)$-module structure on V. Let $f \in \mathcal{H}(G)$ and $v \in V$. The function $x \mapsto f(x)\pi(x)v$ lies in $C_c^\infty(G, V)$ and hence the element

$$\pi(f)v := \int_G f(x)\pi(x)v \, \mathrm{d}\mu_G(x) \in V \tag{2.7}$$

is well-defined. More concretely, we find a compact open subgroup $K \subseteq G$ such that $v \in V^K$ and $f(gk) = f(g)$, for all $g \in G$, $k \in K$. Then $f = \sum_{g \in G/K} f(g)\mathbf{1}_{gK}$, and then

$$\pi(f)v = \sum_{g \in G/K} f(g) \cdot \pi(\mathbf{1}_{gK})v = \mathrm{vol}(K) \sum_{g \in G/K} f(g)\pi(g)v. \tag{2.8}$$

This also shows that $v \in V^K$ if and only if $\pi(e_K)v = v$.

For all $f, f' \in \mathcal{H}(G)$ and $v \in V$ we verify that $\pi(f * f')v = \pi(f)\big(\pi(f')v\big)$. The formula is $\mathbb{C}$-linear in f and f' and hence by Lemma 6.3 we reduce to $f' = \mathbf{1}_{gK}$ and $f = \mathbf{1}_{hU}$, where $U, K \subseteq G$ are compact open subgroups with $U \subseteq gKg^{-1}$. Hence, we have to show

$$\pi\big(\mathbf{1}_{hU} * \mathbf{1}_{gK}\big)v = \pi(\mathbf{1}_{hU})\big(\pi(\mathbf{1}_{gK})v\big).$$

Let $\gamma \in G$. Then $x \mapsto \mathbf{1}_{hU}(x)\mathbf{1}_{gK}(x^{-1}\gamma)$ is the characteristic function $\mathbf{1}_{hU\cap\gamma Kg^{-1}}$. Using $g^{-1}Ug \subseteq K$, we deduce

$$hU \cap \gamma K g^{-1} = \begin{cases} hU, & \text{if } \gamma \in hUgK = hgK; \\ \varnothing, & \text{otherwise.} \end{cases}$$

Now, $(\mathbf{1}_{hU} * \mathbf{1}_{gK})(\gamma) = \int_G \mathbf{1}_{hU\cap\gamma Kg^{-1}}(x)\,\mathrm{d}\mu_G(x) = \mathrm{vol}(U)\mathbf{1}_{hgK}(\gamma)$. We compute

$$\begin{aligned} \pi\big(\mathbf{1}_{hU} * \mathbf{1}_{gK}\big)v &= \mathrm{vol}(U)\cdot\pi(\mathbf{1}_{hgK})v = \mathrm{vol}(U)\,\mathrm{vol}(K)\cdot\pi(hg)v \\ &= \mathrm{vol}(U)\,\mathrm{vol}(K)\cdot\pi(h)\pi(g)v = \mathrm{vol}(U)\pi(h)\big(\pi(\mathbf{1}_{gK})v\big) \\ &= \pi(\mathbf{1}_{hU})\big(\pi(\mathbf{1}_{gK})v\big). \end{aligned}$$

Hence V is a smooth $\mathcal{H}(G)$-module.

If $\varphi\colon V \to W$ is a G-equivariant map, it follows from (2.8) that φ is also $\mathcal{H}(G)$-linear.

Step 2: Let V be a smooth $\mathcal{H}(G)$-module. We construct on V the structure of a smooth G-representation. We first claim that the map

$$\begin{aligned} \mathcal{H}(G) \otimes_{\mathcal{H}(G)} V &\xrightarrow{\cong} V, \\ f \otimes v &\longmapsto f * v \end{aligned} \tag{2.9}$$

is an $\mathcal{H}(G)$-linear isomorphism. It is clearly well-defined and $\mathcal{H}(G)$-linear. Surjectivity follows from smoothness. To prove injectivity, let $f_1, \dots, f_n \in \mathcal{H}(G)$ and $v_1, \dots, v_n \in V$ such that $\sum_{i=1}^n f_i * v_i = 0$. By Proposition 7.4(b) we find an idempotent $e_K \in \mathcal{H}(G)$ such that $f_i = e_K * f_i$, for all i. Then

$$\sum_{i=1}^n f_i \otimes v_i = \sum_{i=1}^n \big(e_K * f_i \otimes v_i\big) = \sum_{i=1}^n \big(e_K \otimes f_i * v_i\big) = e_K \otimes \sum_{i=1}^n f_i * v_i = 0,$$

which shows that (2.9) is injective.

By Lemma 7.3(b) the space $\mathcal{H}(G) \otimes_{\mathcal{H}(G)} V$ is a smooth G-representation via $g \cdot (f \otimes v) = \big(\lambda(g)f\big) \otimes v$. This induces on V the structure of a smooth G-representation. Concretely, if $v \in V$, we choose a compact open subgroup $K \subseteq G$ with $e_K * v = v$ and then

$$\pi(g)v = e_{gK} * v. \tag{2.10}$$

If $\varphi\colon V \to W$ is a $\mathcal{H}(G)$-linear map, it follows from (2.10) that φ is G-equivariant.

Step 3: It remains to show that these actions determine each other. If (V, π) is a G-representation, denote (V, τ) the G-representation obtained from V regarded as a $\mathcal{H}(G)$-module. For each $g \in G$ we have $\tau(g)v = \pi(e_{gK})v = \pi(g)v$, where $K \subseteq G$ is a compact open subgroup with $v \in V^K$.

Conversely, let (V,π) be a smooth $\mathcal{H}(G)$-module and denote (V,τ) the $\mathcal{H}(G)$-module obtained from V regarded as a G-representation. Let $f \in \mathcal{H}(G)$ and $v \in V$. We have to show $\tau(f)*v = \pi(f)*v$. By Lemma 6.3, we reduce to the case where f is of the form $\mathbf{1}_{gK}$ with $v \in V^K$. By (2.8) and (2.10), we have $\tau(\mathbf{1}_{gK})v = \mathrm{vol}(K;\mu_G)\cdot\pi(g)v = \pi(\mathbf{1}_{gK})v$. This finishes the proof. □

Lemma 7.8. *Let $(V,\pi) \in \mathrm{Rep}(G)$ and let $K \subseteq G$ be a compact open subgroup. Then $\pi(e_K)\colon V \to V$ is a K-equivariant projection with image V^K and kernel*

$$\mathrm{Ker}(\pi(e_K)) = V(K) := \big\langle v - \pi(k)v \mid v \in V, k \in K \big\rangle,$$

where $\langle \cdots \rangle$ denotes the $\mathbb{C}$-linear span; in particular, $V \cong V^K \oplus V(K)$ as K-representations. Moreover, $V^K = \pi(e_K)V$ is a unital $\mathcal{H}(G,K)$-module.

Proof. The fact that $\pi(e_K)$ is a projection follows from Proposition 7.4(a). It is clear that $\pi(e_K)_{|V^K}$ is the identity. Let $v \in V$ and $k \in K$. Let $H \subseteq K$ be an open normal subgroup with $v \in V^H$. Then $\pi(e_K)\pi(k)v = \frac{1}{[K:H]}\sum_{x\in K/H}\pi(xk)v = \frac{1}{[K:H]}\sum_{x\in K/H}\pi(x)v = \pi(e_K)v$, and it follows that $V(K) \subseteq \mathrm{Ker}(\pi(e_K))$ and that $\pi(e_K)$ is K-equivariant. Conversely, if $v \in \mathrm{Ker}(\pi(e_K))$, then

$$v = v - \pi(e_K)v = v - \frac{1}{[K:H]}\sum_{k\in K/H}\pi(k)v = \frac{1}{[K:H]}\sum_{k\in K/H}\big(v-\pi(k)v\big) \in V(K).$$

Hence, $V(K) = \mathrm{Ker}(\pi(e_K))$. The other assertions are clear. □

We will now relate the irreducible smooth G-representations with the simple modules of the Hecke algebras $\mathcal{H}(G,K)$.

Theorem 7.9. *Let $K \subseteq G$ be a compact open subgroup.*

(a) Let $(V,\pi) \in \mathrm{Rep}(G)$ be irreducible. Then V^K is either zero or a simple $\mathcal{H}(G,K)$-module.

(b) We have a bijection

$$\left\{\begin{matrix}\text{isomorphism classes of}\\ \text{irreducible } (V,\pi)\in\mathrm{Rep}(G)\\ \text{with } V^K \neq \{0\}\end{matrix}\right\} \xrightarrow{\ \cong\ } \left\{\begin{matrix}\text{isomorphism classes of}\\ \text{simple } \mathcal{H}(G,K)\text{-modules}\end{matrix}\right\},$$
$$(V,\pi) \longmapsto V^K = \pi(e_K)V.$$

Proof. We first prove (a). Let $M \subseteq V^K$ be a non-zero $\mathcal{H}(G,K)$-submodule. As V is irreducible, we have $M \supseteq \pi\big(\mathcal{H}(G,K)\big)M = \pi(e_K)\pi\big(\mathcal{H}(G)\big)M = \pi(e_K)V = V^K$. Hence, V^K is simple.

We have shown that the map in (b) is well-defined. We describe the inverse map. Let M be a simple $\mathcal{H}(G,K)$-module. Consider the smooth G-representation

$$W := \mathcal{H}(G) * e_K \otimes_{\mathcal{H}(G,K)} M.$$

Then $W^K = \pi(e_K)W = e_K * \mathcal{H}(G) * e_K \otimes_{\mathcal{H}(G,K)} M \cong M$.

Let $X(W) \subseteq W$ be the sum of all G-invariant subspaces $X \subseteq W$ with $X^K = \{0\}$. Let $X, Y \subseteq W$ be G-invariant subspaces with $X^K = Y^K = \{0\}$ and consider the surjection $X \oplus Y \twoheadrightarrow X + Y$. Lemma 5.8 shows that the map

$$\{0\} = X^K \oplus Y^K = (X\oplus Y)^K \longrightarrow\!\!\!\!\!\rightarrow (X+Y)^K$$

is surjective. Hence, $(X+Y)^K=\{0\}$. Therefore, $X(W)\subseteq W$ is the largest G-invariant subspace with $X(W)^K=\{0\}$. We claim that $t(M):=W/X(W)$ is irreducible. If $X(W)\subsetneq U\subseteq W$ is a G-invariant subspace, then $U^K\neq\{0\}$ is a $\mathcal{H}(G,K)$-submodule of M. As M is simple, we have $U^K=M$ and hence $W=\mathcal{H}(G)*e_K\otimes_{\mathcal{H}(G,K)}U^K\subseteq U$. Hence, $t(M)$ is irreducible. Again by Lemma 5.8, we have $t(M)^K=W^K/X(W)^K=W^K=M$.

We need to show that the map $[M]\mapsto[t(M)]$ is well-defined. Let $f\colon M\xrightarrow{\cong}M'$ be an $\mathcal{H}(G,K)$-linear isomorphism, we obtain a G-equivariant isomorphism

$$f\colon W=\mathcal{H}(G)*e_K\otimes_{\mathcal{H}(G,K)}M\xrightarrow{\cong}\mathcal{H}(G)*e_K\otimes_{\mathcal{H}(G,K)}M'=:W'$$

such that $f(X(W))=X(W')$. Hence, we obtain an isomorphism $t(M)\xrightarrow{\cong}t(M')$.

It remains to prove injectivity of the map in (b). Let $(V,\pi)\in\mathrm{Rep}(G)$ be irreducible. The inclusion $V^K\subseteq V$ induces a non-zero map $f\colon W:=\mathcal{H}(G)*e_K\otimes_{\mathcal{H}(G,K)}V^K\to V$. Since $f(X(W))\subseteq V$ is a G-invariant subspace with $f(X(W))^K=\pi(e_K)f(X(W))=f(\pi(e_K)X(W))=f(X(W)^K)=\{0\}$ and V is irreducible, we deduce $f(X(W))=\{0\}$. Consequently, f factors through a non-zero map

$$f\colon t(V^K)\to V.$$

Since both $t(V^K)$ and V are irreducible, we have $\mathrm{Ker}(f)=\{0\}$ and $\mathrm{Im}(f)=V$, hence f is an isomorphism. □

Exercise. Let $(V,\pi)\in\mathrm{Rep}(G)$ be a non-zero representation. Show that (V,π) is irreducible if and only if for each compact open subgroup $K\subseteq G$, the space V^K is either zero or a simple $\mathcal{H}(G,K)$-module.

Definition 7.10. We say G is *countable at infinity* if for some (equivalently, for each) compact open subgroup $K\subseteq G$ the set G/K is countable.

The next result shows that the Hecke algebra $\mathcal{H}(G)$ behaves like a semisimple algebra. This will be made more precise in §11.

Theorem 7.11 (Separation Lemma)**.** *Suppose G is countable at infinity. Let $f\in\mathcal{H}(G)$ with $f\neq 0$. There exists an irreducible smooth G-representation (V,π) such that $\pi(f)\neq 0$.*

Proof. Fix a compact open subgroup $K\subseteq G$ with $f=e_K*f*e_K\in\mathcal{H}(G,K)$. Define $f^\dagger\in\mathcal{H}(G,K)$ by $f^\dagger(g):=\overline{f(g^{-1})}$, where the overline means complex conjugation. We have

$$(f^\dagger*f)(1)=\int_G|f(x^{-1})|^2\,\mathrm{d}\mu_G(x)\neq 0.$$

Hence $h:=f^\dagger*f\neq 0$, and for each $g\in G$ we compute

$$\begin{aligned} h^\dagger(g)=\overline{(f^\dagger*f)(g^{-1})}&=\overline{\int_G f^\dagger(x)\cdot f(x^{-1}g^{-1})\,\mathrm{d}\mu_G(x)}\\ &=\int_G f(x^{-1})\cdot\overline{f(x^{-1}g^{-1})}\,\mathrm{d}\mu_G(x)=\int_G\overline{f\big((gx)^{-1}\big)}\cdot f(x^{-1})\,\mathrm{d}\mu_G(x)\\ &=\int_G f^\dagger(gx)\cdot f(x^{-1})\,\mathrm{d}\mu_G(x)=(f^\dagger*f)(g)=h(g);\end{aligned}$$

thus, $h^\dagger = h$. By induction we see $h^{2^n} = (h^\dagger * h)^{2^{n-1}} \neq 0$ for all n; hence $h \in \mathcal{H}(G,K)$ is not nilpotent. Since G is countable at infinity, it follows that $\mathcal{H}(G,K) \subseteq C_c^\infty(G/K)$ has countable dimension over $\mathbb{C}$ (Proposition 7.4 and Lemma 6.3). The assertion now follows from the next lemma. □

Lemma 7.12. *Let R be an associative unital $\mathbb{C}$-algebra of countable dimension and let $h \in R$ be a non-nilpotent element.*

(a) There exists $a \in \mathbb{C}^\times$ such that $R(h-a) \subsetneq R$.

(b) There exists a simple R-module M with $hM \neq \{0\}$.

Proof. We prove (a). If $h \in \mathbb{C}$, then $a = h$ is as desired. Otherwise, we assume for a contradiction that $R(h-a) = R$ for all $a \in \mathbb{C}^\times$. Then the uncountable family $\{1/(h-a) \mid a \in \mathbb{C}^\times\}$ is linearly dependent, since R has countable dimension. Hence, there exist $b_1, \dots, b_n \in \mathbb{C}^\times$ and pairwise distinct $a_1, \dots, a_n \in \mathbb{C}^\times$ such that $\sum_{i=1}^n b_i \cdot 1/(h-a_i) = 0$. Multiplying from the right by $\prod_i (h-a_i)$, we obtain a non-zero polynomial $P(t) \in \mathbb{C}[t]$ with $P(h) = 0$. As $\mathbb{C}$ is algebraically closed, we can write $0 = P(h) = h^{n_0} \prod_j (h-c_j)^{n_j}$, for certain $c_j \in \mathbb{C}^\times$ and $n_0, n_j \in \mathbb{Z}_{\geqslant 1}$. As h is not nilpotent, it follows that one of the factors $h - c_j$ is a (left) zero-divisor, hence $R(h-c_j) \neq R$, which contradicts our assumption.

We prove (b). By (a) there exists $a \in \mathbb{C}^\times$ such that $R(h-a)$ is a proper left ideal in R. By Zorn's lemma there exists a maximal left ideal $\mathfrak{m} \subseteq R$ containing $h - a$. For $M \coloneqq R/\mathfrak{m}$, we then have $hM = aM = M \neq 0$. □

§8. Smooth Representations of Profinite Groups

Let K be a profinite group. In this section we give a precise description of the category $\mathrm{Rep}(K)$ of smooth K-representations. We start with a general result.

Proposition 8.1. *Let G be a group. For $V \in \mathrm{Mod}(\mathbb{C}[G])$, the following are equivalent:*

(i) There exists a family $\{W_i\}_{i\in I}$ of irreducible subrepresentations of V such that $V = \sum_{i\in I} W_i$.

(ii) There exists a family $\{W_i\}_{i\in I}$ of irreducible G-representations with $V \cong \bigoplus_{i\in I} W_i$.

(iii) For every G-invariant subspace $W \subseteq V$, there exists a G-invariant subspace $W' \subseteq V$ with $V = W \oplus W'$.

If these conditions are satisfied, we call V semisimple.

Proof. We show that (i) implies (ii) and (iii). Let $W \subsetneq V$ be a proper G-invariant subspace and write $V = \sum_{i\in I} W_i$ as in (i). The set

$$X \coloneqq \Big\{ J \subseteq I \;\Big|\; W + \sum_{j\in J} W_j \text{ is a direct sum} \Big\}$$

is partially ordered with respect to inclusion and non-empty, since $\varnothing \in X$. Let $Y \subseteq X$ be a totally ordered subset and put $J_0 \coloneqq \bigcup_{J\in Y} J$. We claim $J_0 \in X$, that is, $W + \sum_{j\in J_0} W_j$ is a direct sum. We have to show that the obvious map $\alpha\colon W \oplus \bigoplus_{j\in J_0} W_j \to W + \sum_{j\in J_0} W_j$ is injective. Pick any $w \in \mathrm{Ker}(\alpha)$. Since Y is totally ordered, we have $w \in W \oplus \bigoplus_{j\in J} W_j$ for some $J \in Y$. But since

$J \in X$, this shows $w = 0$ and hence α is injective. We have shown that the upper bound J_0 of Y is contained in X.

Hence, Zorn's Lemma applies and gives a maximal element $J \in X$. Put $V' := W + \sum_{j \in J} W_j \subseteq V$. Take any $i \in I \smallsetminus J$. As W_i is irreducible, we have either $W_i \cap V' = \{0\}$ or $W_i \cap V' = W_i$. In the first case, $W + \sum_{j \in J \cup \{i\}} W_j$ is direct and hence $J \cup \{i\} \in X$, which contradicts the maximality of J. Hence, we must have $W_i \subseteq V'$. As $i \in I \smallsetminus J$ was arbitrary, we conclude $V = \sum_{i \in I} W_i \subseteq V'$. This shows that $W' := \sum_{j \in J} W_j$ is a G-invariant complement of W, whence (iii). The particular case $W = \{0\}$ proves (ii).

The implication "(ii) $\Longrightarrow$ (i)" is trivial. It remains to prove "(iii) $\Longrightarrow$ (i)". Let $V' = \sum_{i \in I} W_i$ be the sum of all irreducible subrepresentations of V. Assume for a contradiction that $V' \subsetneqq V$. By assumption, there exists a G-invariant subspace $V'' \subseteq V$ with $V' \oplus V'' = V$. Let $v \in V'' \smallsetminus \{0\}$ and let $\mathfrak{a} \subsetneqq \mathbb{C}[G]$ be the kernel of the orbit map $\phi \colon \mathbb{C}[G] \to V''$, $f \mapsto fv$. By Zorn's Lemma, there exists a maximal left ideal $\mathfrak{m} \subsetneq \mathbb{C}[G]$ with $\mathfrak{a} \subseteq \mathfrak{m}$. By (iii), there exists a G-invariant subspace $U \subseteq V$ with $V' \oplus \mathfrak{m}v \oplus U = V$. The kernel of the composite map $\mathbb{C}[G] \xrightarrow{\phi} V \xrightarrow{\mathrm{pr}_U} U$ is $\mathfrak{m}$. Hence U contains the irreducible subrepresentation $\mathbb{C}[G]/\mathfrak{m}$. But then $U \cap V' \neq \{0\}$ by the definition of V', which contradicts $U \cap V' = \{0\}$. Hence, the assumption was wrong and we have $V' = V$. □

Exercise 8.2. Let G be a group and let $V \in \mathrm{Mod}(\mathbb{C}[G])$ be semisimple. Show that for every G-invariant subspace $W \subseteq V$ one has that W and V/W are semisimple.

Proposition 8.3. *Let G be a group and $H \subseteq G$ a subgroup of finite index. Let $(V, \pi) \in \mathrm{Mod}(\mathbb{C}[G])$. Then (V, π) is semisimple if and only if $(V, \pi_{|H})$ is semisimple.*

Proof. Step 1: Suppose that $(V, \pi_{|H})$ is semisimple. Let $W \subseteq V$ be a G-invariant subspace. By assumption, there exists an H-invariant subspace $W' \subseteq V$ such that $V = W \oplus W'$. Denote $f' \colon V \to W$ the corresponding H-equivariant projection. The map

$$\begin{aligned} f \colon V &\longrightarrow W, \\ v &\longmapsto \frac{1}{[G:H]} \cdot \sum_{g \in G/H} g f'(g^{-1}v) \end{aligned}$$

is G-equivariant and the identity on W. Hence, $\mathrm{Ker}(f)$ is G-invariant, and $V = W \oplus \mathrm{Ker}(f)$. By Proposition 8.1, (V, π) is semisimple.

Step 2: Suppose (V, π) is semisimple. The subgroup $H_0 := \bigcap_{g \in G/H} gHg^{-1} \subseteq G$ is normal and has finite index, because the canonical map $G/H_0 \to \prod_{g \in G/H} G/gHg^{-1}$ is injective. It suffices to show that $(V, \pi_{|H_0})$ is semisimple, because then also $(V, \pi_{|H})$ is semisimple by Step 1. Without loss of generality, we may assume that (V, π) is irreducible. As $[G : H_0]$ is finite, $(V, \pi_{|H_0})$ is finitely generated as a $\mathbb{C}[H_0]$-module. By Zorn's Lemma, there exists an H_0-equivariant surjection $\phi \colon (V, \pi_{|H_0}) \twoheadrightarrow (U, \sigma)$ onto an irreducible H_0-representation (U, σ). For any $g \in G$, define the H_0-representation $(U, g_*\sigma)$ by $g_*\sigma(h) := \sigma(g^{-1}hg)$, which is clearly irreducible. Fix a representing system $g_1, \ldots, g_r \in G$ of G/H_0. Observe that (E, τ), given by $E := \mathbb{C}[G] \otimes_{\mathbb{C}[H_0]} U$ and $\tau(g)(f \otimes u) := e_g f \otimes u$, is a G-representation and that $(E, \tau_{|H_0}) \cong \bigoplus_{i=1}^r (U, g_{i*}\sigma)$. The map

$$\begin{aligned} \Phi \colon (V, \pi) &\longrightarrow (E, \tau), \\ v &\longmapsto \sum_{i=1}^r e_{g_i} \otimes \phi\big(\pi(g_i^{-1})v\big) \end{aligned}$$

is non-zero. We verify that it is G-equivariant. Let $g \in G$. For each i, there exist unique $1 \leqslant j(i) \leqslant r$ and $h_i \in H$ with $g^{-1}g_i = g_{j(i)}h$. Note that the map $i \mapsto j(i)$ is bijective. Hence, we compute

$$\begin{aligned}\Phi(\pi(g)v) &= \sum_{i=1}^{r} e_{g_i} \otimes \phi\big(\pi(g_i^{-1})\pi(g)v\big) = \sum_{i=1}^{r} e_g e_{g^{-1}g_i} \otimes \phi\big(\pi(g^{-1}g_i)^{-1}v\big)\\ &= \tau(g)\sum_{i=1}^{r} e_{g_{j(i)}} e_h \otimes \phi\big(\pi(h^{-1})\pi(g_{j(i)}^{-1})v\big) = \tau(g)\sum_{i=1}^{r} e_{g_{j(i)}} \otimes \phi\big(\pi(g_{j(i)}^{-1})v\big)\\ &= \tau(g)\Phi(v).\end{aligned}$$

Hence, the map Φ is G-invariant. As V is irreducible, we deduce that Φ is injective. Hence, $(V, \pi_{|H_0})$ is isomorphic to a subrepresentation of the semisimple H_0-representation $\bigoplus_{i=1}^{r}(U, g_{i*}\sigma)$, hence itself semisimple by Exercise 8.2. □

Example 8.4 (Maschke's Theorem)**.** Let G be a finite group. Then every G-representation is semisimple by Proposition 8.3 (for $H = \{1\}$).

Let now K be a profinite group.

Theorem 8.5. *(a) Every irreducible $V \in \mathrm{Rep}(K)$ has finite dimension over $\mathbb{C}$.*

(b) For every finite dimensional smooth K-representation V there exists an open normal subgroup $N \subseteq K$ such that $V = V^N$.

(c) Every $V \in \mathrm{Rep}(K)$ is semisimple.

Proof. (a) was proved in Lemma 5.7. For (b), we pick a basis $v_1, \ldots, v_n$ of V together with open subgroups K_i of K with $v_i \in V^{K_i}$. Then any open normal subgroup $N \subseteq K$ satisfying $N \subseteq \bigcap_{i=1}^{n} K_i$ is as desired.

We prove (c). We show that each $v \in V$ is contained in a semisimple subrepresentation of V. Pick an open normal subgroup $H \subseteq K$ with $v \in V^H$. Then $W := \sum_{k \in K/H} \mathbb{C}kv \subseteq V^H$ is a representation of the finite group K/H, hence is semisimple by Maschke's Theorem. Thus, W is a semisimple K-representation. □

Schur's Lemma 8.6. *Let (V, τ) be an irreducible smooth K-representation. Then*

$$\mathrm{End}_K(V) \cong \mathbb{C}.$$

Proof. Let $\varphi\colon V \to V$ a non-zero K-equivariant map. Since V is finite dimensional by Lemma 5.7 and $\mathbb{C}$ is algebraically closed, φ has an eigenvalue $a \in \mathbb{C}$. Now, the kernel of the map $\varphi - a\,\mathrm{id}_V : V \to V$ is a non-zero K-invariant subspace of V. As V is irreducible, it follows that $\varphi - a\,\mathrm{id}_V = 0$. □

Theorem 8.7. *Let $\mathrm{Vect}_{\mathbb{C}}$ be the category of $\mathbb{C}$-vector spaces. Let $\prod_{\tau \in \mathbf{Irr}(K)} \mathrm{Vect}_{\mathbb{C}}$ be the category whose objects are tuples $(V_\tau)_\tau$ consisting of a $\mathbb{C}$-vector space V_τ for each $\tau \in \mathbf{Irr}(K)$. A morphism $(V_\tau)_\tau \to (V'_\tau)_\tau$ consists of a tuple $(\varphi_\tau)_\tau$, where each $\varphi_\tau\colon V_\tau \to V'_\tau$ is a $\mathbb{C}$-linear map. The functors*

$$\begin{aligned}\mathcal{A}\colon \textstyle\prod_{(V_\tau,\tau)\in\mathbf{Irr}(K)} \mathrm{Vect}_{\mathbb{C}} &\overset{\cong}{\longleftrightarrow} \mathrm{Rep}(K) : \mathcal{F},\\ (W_\tau)_\tau &\longmapsto \textstyle\bigoplus_{\tau\in\mathbf{Irr}(K)} W_\tau \otimes_{\mathbb{C}} V_\tau\\ \big(\mathrm{Hom}_K(V_\tau, V)\big)_\tau &\longleftarrow\!\shortmid V\end{aligned}$$

where K acts on the second factor of $W_\tau \otimes_{\mathbb{C}} V_\tau$, are quasi-inverse equivalences of categories.

Proof. Let $(V,\pi) \in \mathrm{Rep}(K)$. For each $(V_\tau,\tau) \in \mathbf{Irr}(K)$ we let $V(\tau)$ be the sum of all irreducible subrepresentations of V which are isomorphic to τ; we call $V(\tau)$ the *τ-isotypic component* of V. Note that the map

$$\begin{aligned} \mathrm{Hom}_K(V_\tau, V) \otimes_{\mathbb{C}} V_\tau &\xrightarrow{\cong} V(\tau), \\ \varphi \otimes v &\longmapsto \varphi(v), \end{aligned} \tag{2.11}$$

is an isomorphism: Every K-equivariant map $V_\tau \to V$ factors through $V(\tau)$; hence $\mathrm{Hom}_K(V_\tau, V) = \mathrm{Hom}_K(V_\tau, V(\tau))$. Write $V(\tau) = \bigoplus_I V_\tau$ for some set I. We have isomorphisms

$$\begin{aligned} \mathrm{Hom}_K(V_\tau, V) \otimes_{\mathbb{C}} V_\tau &= \mathrm{Hom}_K\big(V_\tau, V(\tau)\big) \otimes_{\mathbb{C}} V_\tau \\ &\cong \bigoplus_I \mathrm{Hom}_K(V_\tau, V_\tau) \otimes_{\mathbb{C}} V_\tau && (V_\tau \text{ is finitely generated}) \\ &\cong \bigoplus_I V_\tau && (\text{Schur's Lemma 8.6}) \\ &\cong V(\tau). \end{aligned}$$

Now check that the composite coincides with (2.11). For the second isomorphism, we have used that V_τ is finitely generated as a K-representation.[3] We obtain a natural isomorphism $\mathcal{A}\big(\mathcal{F}(V)\big) = \bigoplus_{\tau \in \mathbf{Irr}(K)} \mathrm{Hom}_K(V_\tau, V) \otimes_{\mathbb{C}} V_\tau \cong \bigoplus_\tau V(\tau) = V$, where the second equality follows from Theorem 8.5(c).

Let now $(W_\tau)_\tau \in \prod_\tau \mathrm{Vect}_{\mathbb{C}}$. For each $(V_\sigma, \sigma) \in \mathbf{Irr}(K)$ we have $\big(\bigoplus_\tau W_\tau \otimes_{\mathbb{C}} V_\tau\big)(\sigma) = W_\sigma \otimes_{\mathbb{C}} V_\sigma$. By Schur's Lemma 8.6, and since V_σ is finitely generated, we have isomorphisms

$$W_\sigma \cong \mathrm{Hom}_K(V_\sigma, V_\sigma) \otimes_{\mathbb{C}} W_\sigma \cong \mathrm{Hom}_K\big(V_\sigma, W_\sigma \otimes_{\mathbb{C}} V_\sigma\big).$$

Hence, we have a natural isomorphism

$$\begin{aligned} \mathcal{F}\big(\mathcal{A}((W_\tau)_\tau)\big) = \mathcal{F}\Big(\bigoplus_\tau W_\tau \otimes_{\mathbb{C}} V_\tau\Big) &= \Big[\mathrm{Hom}_K\Big(V_\sigma, \bigoplus_\tau W_\tau \otimes_{\mathbb{C}} V_\tau\Big)\Big]_\sigma \\ &= \big[\mathrm{Hom}_K(V_\sigma, W_\sigma \otimes_{\mathbb{C}} V_\sigma)\big]_\sigma \cong \big[W_\sigma\big]_\sigma. \end{aligned}$$

This finishes the proof. □

Remark. Theorem 8.7 makes precise the idea that $\mathrm{Rep}(K)$ is completely determined by the set $\mathbf{Irr}(K)$ of (isomorphism classes of) irreducible smooth K-representations. If G is a locally profinite group, then not every smooth G-representation will be semisimple. Hence, the category $\mathrm{Rep}(G)$ has a lot more structure than $\mathrm{Rep}(K)$.

Our ultimate goal in this lecture will be to prove a decomposition theorem for $\mathrm{Rep}(\mathrm{GL}_n(F))$ when F is a local field.

[3] Given a family of K-representations W_i, $i \in I$, we have a map $\bigoplus_{i\in I} \mathrm{Hom}_K(V_\tau, W_i) \to \mathrm{Hom}_K(V_\tau, \bigoplus_{i \in I} W_i)$, $(\varphi_i)_i \mapsto [v \mapsto \sum_{i \in I} \varphi_i(v)]$. Injectivity is clear. In order to prove surjectivity, let $\varphi \in \mathrm{Hom}_K(V_\tau, \bigoplus_{i\in I} W_i)$. Let $v \in V_\tau \smallsetminus \{0\}$ so that $\varphi(v) \in \bigoplus_{j \in J} W_j$ for some finite subset $J \subseteq I$. As V_τ is irreducible, it is generated by v, hence $\varphi(V_\tau) \subseteq \bigoplus_{j\in J} W_j$. Denoting $\mathrm{pr}_j \colon \bigoplus_{i \in I} W_i \to W_j$ the j-th projection, we deduce that φ is the image of $(\varphi_i)_i$, where $\varphi_i = 0$ for $i \in I \smallsetminus J$ and $\varphi_j = \mathrm{pr}_j \circ \varphi$ for $j \in J$.

Exercise. Let F be a local field. Construct an equivalence of categories

$$\operatorname{Rep}(F^\times) \cong \prod_{\chi \in \mathbf{Irr}(\mathfrak{o}_F^\times)} \operatorname{Mod}\big(\mathbb{C}[t, t^{-1}]\big).$$

§9. Smooth and Compact Induction

Let G be a locally profinite group and let $H \subseteq G$ be a closed subgroup. There is a forgetful functor

$$\operatorname{Res}_H^G \colon \operatorname{Rep}(G) \longrightarrow \operatorname{Rep}(H)$$

defined by $\operatorname{Res}_H^G(V, \pi) = (V, \pi_{|H})$, where $\pi_{|H}$ is the restriction of $\pi \colon G \to \operatorname{Aut}_{\mathbb{C}}(V)$ to H. (We will often just write V or $\operatorname{Res}_H^G V$ or $\pi_{|H}$ instead of $\operatorname{Res}_H^G(V, \pi)$.) In this section we will construct two functors in the other direction, $\operatorname{Rep}(H) \to \operatorname{Rep}(G)$, which allow us to construct new smooth G-representations out of smooth H-representations.

Definition 9.1. Let $(W, \sigma) \in \operatorname{Rep}(H)$ be a smooth H-representation. Put

$$\operatorname{IND}_H^G W := \{f \colon G \to W \mid f(hg) = \sigma(h) f(g) \text{ for all } h \in H,\, g \in G\}.$$

The group G acts on $\operatorname{IND}_H^G W$ via right translation: $(gf)(g') := f(g'g)$ for all $f \in \operatorname{IND}_H^G W$ and $g, g' \in G$. We define $\operatorname{Ind}_H^G W$ as the (G-invariant) subspace of all functions $f \in \operatorname{IND}_H^G W$ which have an open stabilizer. We denote the induced action of G on Ind_H^G by $\operatorname{Ind}_H^G \sigma$. We obtain a functor

$$\operatorname{Ind}_H^G \colon \operatorname{Rep}(H) \longrightarrow \operatorname{Rep}(G),$$

defined by $\operatorname{Ind}_H^G(W, \sigma) := (\operatorname{Ind}_H^G W, \operatorname{Ind}_H^G \sigma)$, which we call *smooth induction*.

Example 9.2. If $W = \mathbb{C}$ is the trivial H-representation, then $\operatorname{Ind}_H^G \mathbb{C} =: C^\infty(H \backslash G)$ is the space of all functions $f \colon G \to \mathbb{C}$ for which there exists a compact open subgroup $K \subseteq G$ such that $f(hgk) = f(g)$ for all $h \in H$, $g \in G$, $k \in K$. These functions are also called *uniformly locally constant*.

Proposition 9.3 (Frobenius Reciprocity)**.** *Let $(V, \pi) \in \operatorname{Rep}(G)$ and $(W, \sigma) \in \operatorname{Rep}(H)$. Consider the H-equivariant homomorphism $\operatorname{Ind}_H^G W \to W$, $f \mapsto f(1)$. Then the canonical map*

$$\begin{aligned} \alpha_* \colon \operatorname{Hom}_G(\pi, \operatorname{Ind}_H^G \sigma) &\xrightarrow{\cong} \operatorname{Hom}_H(\pi_{|H}, \sigma), \\ \phi &\longmapsto \big[v \mapsto \phi(v)(1)\big] \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism, natural in V and W.

Proof. The map is clearly well-defined, $\mathbb{C}$-linear, and natural in V and W. We describe the inverse map. Consider the natural map

$$\begin{aligned} \beta \colon V &\longrightarrow \operatorname{Ind}_H^G V, \\ v &\longmapsto [g \mapsto \pi(g) v]. \end{aligned}$$

Note that $\beta(v)$ lies in $\operatorname{Ind}_H^G V$: Let $K \subseteq G$ be a compact open subgroup with $v \in V^K$. Then $\beta(v)(gk) = \pi(gk)v = \pi(g)\pi(k)v = \pi(g)v = \beta(v)(g)$ for all $g \in G$ and $k \in K$. Moreover, β is G-equivariant, since for all $v \in V$, $g, g' \in G$ we have

$$\big(g\beta(v)\big)(g') = \beta(v)(g'g) = \pi(g'g)v = \pi(g')\big(\pi(g)v\big) = \beta\big(\pi(g)v\big)(g').$$

We claim that the natural map

$$\begin{aligned} \beta^*\colon \operatorname{Hom}_H\big(\pi_{|H}, \sigma\big) &\longrightarrow \operatorname{Hom}_G\big(\pi, \operatorname{Ind}_H^G \sigma\big), \\ \psi &\longmapsto \big[v \mapsto \psi \circ \beta(v)\big] \end{aligned}$$

is inverse to α_*. Let $\psi\colon V \to W$ be H-equivariant. We claim that $\alpha_*(\beta^*(\psi)) = \psi$: Indeed, for each $v \in V$ we compute

$$\alpha_*\big(\beta^*(\psi)\big)(v) = \beta^*(\psi)(v)(1) = \psi\big(\beta(v)(1)\big) = \psi(v).$$

Conversely, let $\phi\colon V \to \operatorname{Ind}_H^G W$ be G-equivariant. We claim that $\beta^*(\alpha_*(\phi)) = \phi$: Indeed, for all $v \in V$ and $g \in G$ we compute

$$\begin{aligned} \big[\beta^*\big(\alpha_*(\phi)\big)(v)\big](g) &= \alpha_*(\phi)\big(\beta(v)(g)\big) = \alpha_*(\phi)\big(\pi(g)v\big) \\ &= \phi\big(\pi(g)v\big)(1) = (g\phi)(v)(1) = \phi(v)(g). \end{aligned}$$

This shows that α_* is an isomorphism. □

Remark. In categorical terms, Proposition 9.3 says that the functor Ind_H^G is *right adjoint* to Res_H^G (or that Res_H^G is *left adjoint* to Ind_H^G). We will later show that, if $H \subseteq G$ is open, then Res_H^G also admits a left adjoint.

Exercise 9.4. Let G be a locally profinite group and $N \trianglelefteq G$ a closed normal subgroup. Denote $\varphi\colon G \twoheadrightarrow G/N$ the projection. For $(W, \sigma) \in \operatorname{Rep}(G/N)$ we write $\operatorname{Inf}_G^{G/N} \sigma = \sigma \circ \varphi\colon G \to \operatorname{Aut}_{\mathbb{C}}(W)$. We obtain a smooth representation $\operatorname{Inf}_G^{G/N}(W, \sigma) := (W, \operatorname{Inf}_G^{G/N} \sigma) \in \operatorname{Rep}(G)$. Let $(V, \pi) \in \operatorname{Rep}(G)$.

(a) Show that G/N naturally acts on V^N and that it yields a smooth representation $(V^N, \pi^N) \in \operatorname{Rep}(G/N)$. Construct a natural $\mathbb{C}$-linear bijection

$$\operatorname{Hom}_G\big(\operatorname{Inf}_G^{G/N} \sigma, \pi\big) \xrightarrow{\cong} \operatorname{Hom}_{G/N}\big(\sigma, \pi^N\big).$$

Hence, $\operatorname{Inf}_G^{G/N}$ is left adjoint to $\pi \mapsto \pi^N$. Informally, this means that V^N is the biggest subspace of V on which N acts trivially.

(b) Show that G/N naturally acts on $V_N := V/V(N)$, where $V(N) = \langle v - \pi(n)v \mid v \in V, n \in N\rangle$, and that it yields a smooth representation $(V_N, J_N(\pi)) \in \operatorname{Rep}(G/N)$. Construct a natural $\mathbb{C}$-linear bijection

$$\operatorname{Hom}_G\big(\pi, \operatorname{Inf}_G^{G/N} \sigma\big) \xrightarrow{\cong} \operatorname{Hom}_{G/N}\big(J_N(\pi), \sigma\big).$$

Hence, $\operatorname{Inf}_G^{G/N}$ is right adjoint to J_N, which is called the *Jacquet functor*. Informally, this means that V_N is the biggest quotient of V on which N acts trivially.

Proposition 9.5 (Mackey decomposition)**.** *Let K be an open and H a closed subgroup of G. Let $(W,\sigma) \in \mathrm{Rep}(H)$. For each $g \in G$ denote $(W, g_*^{-1}\sigma) \in \mathrm{Rep}(g^{-1}Hg)$ the representation given by $(g_*^{-1}\sigma)(x) := \sigma(gxg^{-1})$ for each $x \in g^{-1}Hg$. The map*

$$\begin{aligned} \mathrm{Res}_K^G \mathrm{Ind}_H^G \sigma &\xrightarrow{\cong} \Big(\prod_{g \in H\backslash G/K} \mathrm{Ind}_{g^{-1}Hg\cap K}^K g_*^{-1}(\sigma_{|H\cap gKg^{-1}}) \Big)^\infty, \\ f &\longmapsto (f_g)_g, \qquad \text{where } f_g(k) = f(gk), \end{aligned}$$

is a K-equivariant isomorphism.

Proof. Since the double cosets HgK are open in G, we have a K-equivariant isomorphism

$$\begin{aligned} \mathrm{Res}_K^G \mathrm{Ind}_H^G \sigma &\xrightarrow{\cong} \Big(\prod_{g \in H\backslash G/K} \mathrm{Ind}_H^{HgK} \sigma \Big)^\infty, \\ f &\longmapsto \big(f_{|HgK}\big)_g. \end{aligned}$$

Hence, for each fixed $g \in G$, we have to show that the map

$$\begin{aligned} \mathrm{Ind}_H^{HgK} \sigma &\xrightarrow{\cong} \mathrm{Ind}_{g^{-1}Hg\cap K}^K g_*^{-1}\sigma_{|H\cap gKg^{-1}}, \\ f &\longmapsto [k \mapsto f(gk)] = f_g \end{aligned} \tag{2.12}$$

is a K-equivariant isomorphism. Note that f_g is well-defined: Let $k \in K$ and $x \in g^{-1}Hg \cap K$. Then

$$f_g(xk) = f(gxk) = f(gxg^{-1}gk) = \sigma(gxg^{-1})f(gk) = \big(g_*^{-1}\sigma_{|H\cap gKg^{-1}}\big)(x) f_g(k).$$

As K acts by right translation, it is clear that $f \mapsto f_g$ is K-equivariant. The inverse map is given by

$$f' \mapsto \widehat{f'} := \big[hgk \mapsto \sigma(h)f'(k)\big].$$

Again, $\widehat{f'}$ is well-defined: Let $h, h' \in H$ and $k, k' \in K$ with $hgk = h'gk'$. Then $x := kk'^{-1} = g^{-1}h^{-1}h'g \in g^{-1}Hg \cap K$. We deduce $xk' = k$ and $hgxg^{-1} = h'$ and compute

$$\begin{aligned} \widehat{f'}(hgk) &= \sigma(h)f'(k) = \sigma(h)f'(xk') = \sigma(h)g_*^{-1}\sigma_{|H\cap gKg^{-1}}(x)f'(k') \\ &= \sigma(h)\sigma(gxg^{-1})f'(k') = \sigma(hgxg^{-1})f'(k') = \sigma(h')f'(k'). \end{aligned}$$

Finally, we check $\widehat{f_g}(hgk) = \sigma(h)f_g(k) = \sigma(h)f(gk) = f(hgk)$ and $(\widehat{f'})_g(k) = \widehat{f'}(gk) = f'(k)$ for all $k \in K$ and $h \in H$. Hence, the maps $f \mapsto f_g$ and $f' \mapsto \widehat{f'}$ are indeed inverse to each other. ☐

Definition 9.6. Let $(W,\sigma) \in \mathrm{Rep}(H)$. The subspace

$$\mathrm{ind}_H^G W := \Big\{ f \in \mathrm{Ind}_H^G W \,\Big|\, \text{the image of } \mathrm{Supp}(f) \text{ in } H\backslash G \text{ is compact} \Big\} \subseteq \mathrm{Ind}_H^G W$$

is G-invariant; here, $\mathrm{Supp}(f)$ satisfies $\mathrm{Supp}(gf) = \mathrm{Supp}(f)g^{-1}$ for $g \in G$, and is defined as in Example 5.3(c). We obtain a functor

$$\mathrm{ind}_H^G \colon \mathrm{Rep}(H) \longrightarrow \mathrm{Rep}(G),$$

defined by $\mathrm{ind}_H^G(W,\sigma) := \big(\mathrm{ind}_H^G W, \mathrm{ind}_H^G \sigma\big)$, is called *compact induction*.

Remark. If $H\backslash G$ is compact, then $\operatorname{ind}_H^G W = \operatorname{Ind}_H^G W$.

Exercise 9.7. Recall that an additive functor $\mathcal{F}\colon \operatorname{Rep}(H) \to \operatorname{Rep}(G)$ is called *exact* if for all H-equivariant maps $W' \xrightarrow{\phi} W \xrightarrow{\psi} W''$ with $\operatorname{Ker}(\psi) = \operatorname{Im}(\phi)$ the induced maps

$$\mathcal{F}(W') \xrightarrow{\mathcal{F}(\phi)} \mathcal{F}(W) \xrightarrow{\mathcal{F}(\psi)} \mathcal{F}(W'')$$

satisfy $\operatorname{Ker}\big(\mathcal{F}(\psi)\big) = \operatorname{Im}\big(\mathcal{F}(\phi)\big)$. Show that the induction functors ind_H^G and Ind_H^G are exact.

Construction 9.8. Suppose $H \subseteq G$ is open, and let $(W, \sigma) \in \operatorname{Rep}(H)$. For all $g \in G$ and $w \in W$ we define $[g, w] \in \operatorname{ind}_H^G W$ via

$$[g,w](x) := \begin{cases} \sigma(xg)w, & \text{if } x \in Hg^{-1}, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $[g, w]$ is the unique function in $\operatorname{ind}_H^G W$ with $\operatorname{Supp}([g, w]) = Hg^{-1}$ and $[g, w](g^{-1}) = w$.

The following properties are immediate:

(i) $[gg', w] = g[g', w]$ for all $g, g' \in G$, $w \in W$;

(ii) $[gh, w] = [g, \sigma(h)w]$ for all $h \in G$, $g \in G$, $w \in W$;

(iii) $f = \sum_{g \in H\backslash G} [g^{-1}, f(g)]$ for all $f \in \operatorname{ind}_H^G W$.[4]

Proposition 9.9 (Frobenius Reciprocity)**.** *Suppose H is open in G. Let $(W, \sigma) \in \operatorname{Rep}(H)$ and $(V, \pi) \in \operatorname{Rep}(G)$. The canonical map*

$$\begin{aligned} \beta^*\colon \operatorname{Hom}_G(\operatorname{ind}_H^G \sigma, \pi) &\xrightarrow{\cong} \operatorname{Hom}_H(\sigma, \pi_{|H}), \\ \psi &\longmapsto \big[w \mapsto \psi([1, w])\big] \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism, natural in V and W.

Proof. The map is clearly well-defined, $\mathbb{C}$-linear, and natural in V and W. We describe the inverse map. Consider the natural map

$$\begin{aligned} \alpha\colon \operatorname{ind}_H^G \operatorname{Res}_H^G(V) &\longrightarrow V, \\ [g, v] &\longmapsto \pi(g)v \end{aligned}$$

and extend by linearity (see (iii) above). It is clear from (i) that α is G-equivariant. We claim that the natural map

$$\begin{aligned} \alpha_*\colon \operatorname{Hom}_H(\sigma, \pi_{|H}) &\longrightarrow \operatorname{Hom}_G(\operatorname{ind}_H^G \sigma, \pi), \\ \phi &\longmapsto \big[[g, w] \mapsto \pi(g)\phi(w)\big] \end{aligned}$$

is inverse to β^*. Let $\phi\colon W \to V$ be H-equivariant. We claim that $\beta^*(\alpha_*(\phi)) = \phi$: Indeed, for each $w \in W$ we compute

$$\beta^*\big(\alpha_*(\phi)\big)(w) = \alpha_*(\phi)\big([1, w]\big) = \pi(1)\phi(w) = \phi(w).$$

[4] Recall: "$g \in H\backslash G$" means that g runs through a set of representatives of $H\backslash G$, and that the sum is finite and independent of this choice.

Conversely, let $\psi\colon \operatorname{ind}_H^G W \to V$ be G-equivariant. We claim that $\alpha_*(\beta^*(\psi)) = \psi$: Indeed, for all $[g,w] \in \operatorname{ind}_H^G W$ we have

$$\alpha_*\big(\beta^*(\psi)\big)([g,w]) = \pi(g)\beta^*(\psi)(w) = \pi(g)\psi\big([1,w]\big) = \psi\big(g[1,w]\big) = \psi\big([g,w]\big).$$

Hence, β^* is an isomorphism. □

Corollary 9.10. *Let $(V,\pi) \in \operatorname{Rep}(G)$ and $K \subseteq G$ a compact open subgroup. Then*

$$\begin{aligned}\operatorname{Hom}_G\big(\operatorname{ind}_K^G \mathbb{C}, V\big) &\xrightarrow{\cong} V^K,\\ \phi &\longmapsto \phi([1,1])\end{aligned}$$

is a $\mathbb{C}$-linear isomorphism.

Proof. By Proposition 9.9 it suffices to show that

$$\begin{aligned}\operatorname{Hom}_K\big(\mathbb{C}, V\big) &\longrightarrow V^K,\\ \phi &\longmapsto \phi(1)\end{aligned}$$

is an isomorphism. If $\phi\colon \mathbb{C} \to V$ is K-equivariant, then $k\phi(1) = \phi(k.1) = \phi(1)$ for all $k \in K$, so that $\phi(1) \in V^K$. The rest is clear. □

§10. The Contragredient and Admissibility

Let G be a locally profinite group. If (V,π) is a smooth G-representation, then the algebraic dual

$$V^* := \operatorname{Hom}_{\mathbb{C}}(V,\mathbb{C})$$

admits a G-action via $(g\varphi)(v) := \varphi(\pi(g^{-1})v)$, for $\varphi \in V^*$, $v \in V$ and $g \in G$. However, the G-representation V^* need not be smooth.

Exercise. Find a locally profinite group G and $V \in \operatorname{Rep}(G)$ such that V^* is not smooth.
(Hint: Realize the example in Exercise 5.5 as an algebraic dual of a smooth $\mathbb{Z}_p$-representation.)

Definition 10.1. Let $(V,\pi) \in \operatorname{Rep}(G)$. Let $\widetilde{V} \subseteq V^*$ be the subspace consisting of all $\mathbb{C}$-linear forms $\varphi\colon V \to \mathbb{C}$ which have an open stabilizer. This defines a smooth G-representation $(\widetilde{V}, \widetilde{\pi})$ called the *contragredient* (or *smooth dual*) representation of (V,π). We then have a canonical pairing

$$\begin{aligned}\langle\,\cdot\,,\cdot\,\rangle\colon \widetilde{V} \times V &\longrightarrow \mathbb{C},\\ (\xi, v) &\longmapsto \langle \xi, v\rangle := \xi(v).\end{aligned} \tag{2.13}$$

Note that $\langle \widetilde{\pi}(g)\xi, \pi(g)v\rangle = \langle \xi, v\rangle$ for all $g \in G$, $\xi \in \widetilde{V}$, and $v \in V$.

Lemma 10.2. *Let $(V,\pi) \in \operatorname{Rep}(G)$ and let $K \subseteq G$ be a compact open subgroup. Then*

$$\widetilde{V}^K = (V^*)^K \cong (V^K)^*.$$

In particular, for all non-zero $v \in V$ there exists $\xi \in \widetilde{V}$ with $\langle \xi, v\rangle \neq 0$.

Proof. Only the isomorphism needs a proof. Let $V(K) = \langle v - \pi(k)v \mid v \in V, k \in K\rangle$. By Lemma 7.8 we have a decomposition

$$V \cong V^K \oplus V(K) \tag{2.14}$$

as K-representations. For $\xi \in V^* = (V^K)^* \oplus V(K)^*$ we have the following equivalences:

$$\begin{aligned} \xi \in (V^*)^K &\iff \xi\big(\pi(k)v\big) = \xi(v), \qquad \text{for all } k \in K,\, v \in V \\ &\iff \xi_{|V(K)} = 0, \\ &\iff \xi \in (V^K)^*. \end{aligned}$$

For the last assertion, let $K \subseteq G$ be a compact open subgroup with $v \in V^K$. Then take any $\xi \in (V^K)^* \subseteq \widetilde{V}$ with $\xi(v) \neq 0$. □

In order to reasonably study smooth representations, we need to impose some finiteness conditions.

Definition 10.3. A smooth G-representation (V, π) is called *admissible* if V^K is finite dimensional for all compact open subgroups $K \subseteq G$.

Exercise. Let $(V, \pi) \in \mathrm{Rep}(G)$ and fix a compact open subgroup $K \subseteq G$. Show that the following are equivalent:

(i) (V, π) is admissible;

(ii) $\mathrm{Hom}_K(\tau, \pi_{|K})$ is finite dimensional, for all $\tau \in \mathbf{Irr}(K)$.

(Hint: For "(i) $\Longrightarrow$ (ii)" use that each $\tau \in \mathbf{Irr}(K)$ becomes trivial after restriction to some open subgroup. For "(ii) $\Longrightarrow$ (i)", decompose $\mathrm{ind}_H^K \mathbb{C}$ into irreducible components, for each open $H \subseteq K$.)

Proposition 10.4. *Let $(V, \pi) \in \mathrm{Rep}(G)$. The following are equivalent:*

(i) (V, π) is admissible;

(ii) $(\widetilde{V}, \widetilde{\pi})$ is admissible;

(iii) The canonical map $V \to \widetilde{\widetilde{V}}$, sending v to the map $[\phi \mapsto \phi(v)]$, is an isomorphism.

Proof. Apply Lemma 10.2. Note that $V \to \widetilde{\widetilde{V}}$ is an isomorphism if and only if for all compact open subgroups $K \subseteq G$ the map $V^K \to \big(\widetilde{\widetilde{V}}\big)^K = (V^K)^{**}$ is bijective. □

Exercise 10.5. (a) Show that the functor $V \mapsto \widetilde{V}$ is exact. (Hint: Use Lemma 5.8.)

(b) Let $(V, \pi) \in \mathrm{Rep}(G)$ be admissible. Show that (V, π) is irreducible if and only if $(\widetilde{V}, \widetilde{\pi})$ is irreducible.

Schur's Lemma 10.6. *Let $(V, \pi) \in \mathrm{Rep}(G)$ be an irreducible representation. Then* $\mathrm{End}_G(V)$ *is a* division algebra.[5]

If, in addition, (V, π) is admissible, then $\mathrm{End}_G(V) \cong \mathbb{C}$.

[5] A division algebra is an associative unital $\mathbb{C}$-algebra D such that every non-zero element of D has a two-sided multiplicative inverse in D.

Proof. Let $\varphi \in \mathrm{End}_G(V)$, $\varphi \neq 0$. Then $\mathrm{Ker}(\varphi) \subsetneq V$ and $\{0\} \neq \mathrm{Im}(\varphi) \subseteq V$ are G-invariant subspaces. As V is irreducible, we have $\mathrm{Ker}(\varphi) = \{0\}$ and $\mathrm{Im}(\varphi) = V$. Hence φ is an isomorphism. This shows that $\mathrm{End}_G(V)$ is a division algebra.

Suppose now that (V, π) is admissible. Choose a compact open subgroup $K \subseteq G$ such that V^K is non-zero. Now, let $\varphi \in \mathrm{End}_G(V)$. As V^K is finite dimensional and $\mathbb{C}$ is algebraically closed, $\varphi_{|V^K}$ admits an eigenvalue, say, $\lambda \in \mathbb{C}$. Then $\varphi - \lambda\,\mathrm{id}_V \in \mathrm{End}_G(V)$ is not an isomorphism and hence $\varphi - \lambda\,\mathrm{id}_V = 0$ by the discussion above. □

Exercise 10.7. Let $(V, \pi) \in \mathrm{Rep}(G)$ be an irreducible admissible representation. Let $B \colon \widetilde{V} \times V \to \mathbb{C}$ be a $\mathbb{C}$-bilinear form such that $B\big(\widetilde{\pi}(g)\xi, \pi(g)v\big) = B(\xi, v)$ for all $g \in G$ and all $v \in V$, $\xi \in \widetilde{V}$.

Show that there exists $a \in \mathbb{C}$ such that $B(\xi, v) = a \cdot \langle \xi, v\rangle$ for all $v \in V$, $\xi \in \widetilde{V}$.

Proposition 10.8. *Let G, H be locally profinite groups and let $(V, \pi) \in \mathrm{Rep}(G)$, $(W, \sigma) \in \mathrm{Rep}(H)$ be irreducible admissible representations. Then $(V \otimes_{\mathbb{C}} W, \pi \otimes \sigma)$ is an irreducible admissible $G \times H$-representation.*

Proof. We first show that $V \otimes_{\mathbb{C}} W$ is admissible. For all compact open subgroups $K \subseteq G$ and $U \subseteq H$, we have

$$\big(V \otimes_{\mathbb{C}} W\big)^{K\times U} = \Big(\big(V \otimes_{\mathbb{C}} W\big)^{K \times \{1\}}\Big)^{\{1\}\times U} = \big(V^K \otimes_{\mathbb{C}} W\big)^{\{1\}\times U} = V^K \otimes_{\mathbb{C}} W^U, \tag{2.15}$$

which is finite dimensional, since V and W are admissible. As every compact open subgroup of $G \times H$ contains a group of the form $K \times U$, it follows that $V \otimes_{\mathbb{C}} W$ is admissible.

To check that $V \otimes_{\mathbb{C}} W$ is irreducible, let $X \subseteq V \otimes_{\mathbb{C}} W$ be a non-zero $G \times H$-invariant subspace. If X contains a simple tensor, say $v \otimes w$, then

$$V \otimes W = \mathbb{C}[G]v \otimes \mathbb{C}[H]w = (\mathbb{C}[G] \otimes \mathbb{C}[H]) \cdot (v \otimes w) \subseteq X$$

shows that $X = V \otimes_{\mathbb{C}} W$. Hence, it suffices to show that X contains a non-zero simple tensor. Let now $x = \sum_{i=1}^n v_i \otimes w_i \in X$, where $n \in \mathbb{Z}_{\geqslant 1}$, $v_1, \ldots, v_n \in V$, and $w_1, \ldots, w_n \in W$. Without loss of generality, we may assume that $v_1, \ldots, v_n$ are $\mathbb{C}$-linearly independent and that $w_n \neq 0$. By Schur's Lemma 10.6, we have $\mathrm{End}_G(V) \cong \mathbb{C}$. We can thus apply Jacobson's Density Theorem 10.9 to obtain $r \in \mathbb{C}[G]$ such that $rv_i = v_i$ for $1 \leqslant i \leqslant n-1$, and $rv_n = 0$. Then $0 \neq v_n \otimes w_n = x - (r \otimes 1)x \in X$ as desired. □

Jacobson's Density Theorem 10.9. *Let R be an associative unital ring, let M be a simple left R-module. Write $D \coloneqq \mathrm{End}_R(M)$.*[6] *Let $x_1, \ldots, x_n \in M$ be linearly independent over D, and let $y_1, \ldots, y_n \in M$ be arbitrary. Then there exists $r \in R$ such that $rx_i = y_i$ for all $i = 1, \ldots, n$.*

Proof. The argument is taken from [Put]. We do an induction on n. Let $n = 1$. Then any $x \in M \smallsetminus \{0\}$ is D-linearly independent, and $Rx = M$, since M is simple. Hence, the statement is clear.

Now, let $n > 1$. We will show the following

Claim. There exist $\lambda_1, \ldots, \lambda_n \in R$ such that $\lambda_i x_i \neq 0$ for all $1 \leqslant i \leqslant n$, and $\lambda_i x_j = 0$ for all $i \neq j$.

[6]Note that D is a division algebra by Schur's Lemma 10.6 and M is a D-module.

One the claim is proven, we argue as follows: By the case $n = 1$, we find $r_i \in R$ such that $r_i \lambda_i x_i = y_i$ for each i. Then $r = \sum_{i=1}^n r_i \lambda_i \in R$ satisfies $r x_i = y_i$ for all i, finishing the proof.

It remains to prove the claim. Fix $1 \leqslant i_0 \leqslant n$. In order to produce a contradiction, we assume that the following property holds:

(P_{i_0}) For all $r \in R$ such that $r x_i = 0$ for all $i \neq i_0$, we have $r x_{i_0} = 0$.

Up to reordering the x_i, we may assume without loss of generality that $i_0 = n$. Define an R-linear map $f\colon M^{n-1} \to M$ as follows: Let $(z_1, \ldots, z_{n-1}) \in M^{n-1}$. By the induction hypothesis, there exists $a \in R$ such that $a x_i = z_i$ for all $1 \leqslant i \leqslant n-1$. We then define

$$f(z_1, \ldots, z_{n-1}) = a x_n.$$

Observe that f is well-defined: If $a' \in R$ is another element with $a' x_i = z_i$ for all $1 \leqslant i \leqslant n-1$, then $(a - a') x_i = 0$ for all $1 \leqslant i \leqslant n-1$ and hence $(a - a') x_n = 0$ by property (P). But this means $a x_n = a' x_n$, so f is indeed well-defined.

For each $1 \leqslant i \leqslant n-1$, we define $\pi_i \in D = \mathrm{End}_R(M)$ as the composition $M \xrightarrow{\iota_i} M^{n-1} \xrightarrow{f} M$, where ι_i is the inclusion of M into the i-th summand. For all $z_1, \ldots, z_{n-1}$ we thus have

$$f(z_1, \ldots, z_{n-1}) = \pi_1 \cdot z_1 + \pi_2 \cdot z_2 + \cdots + \pi_{n-1} \cdot z_{n-1}.$$

In particular, we have $x_n = f(x_1, \ldots, x_{n-1}) = \sum_{i=1}^{n-1} \pi_i x_i$, which contradicts the fact that the x_i are D-linearly independent. Hence, property (P_{i_0}) is not satisfied, so we find $\lambda_n \in R$ as in the claim. □

§11. Compact Representations

In this section we will generalize the results of §8. Let G be a locally profinite group. We will study a class of smooth representations of G which behave like smooth representations of a profinite group.

We fix a left Haar measure μ_G (Definition 6.2). From Theorem 11.7 on we make the assumption that $\mu_G(\rho(g)f) = \mu_G(f)$ for all $g \in G$, $f \in C_c^\infty(G)$; in this case, G is called *unimodular*. By Exercise 6.5, G is unimodular if and only if the modulus character $\delta_G\colon G \to \mathbb{C}^\times$ is trivial.

Example 11.1. (a) If G is compact or, more generally, if G is the union of its compact open subgroups, then G is unimodular.

(b) We will see later (Proposition 12.18) that for any local field F, the group $\mathrm{GL}_n(F)$ is unimodular. But the subgroup B of upper triangular matrices in $\mathrm{GL}_n(F)$ is not unimodular. (Prove this for $n = 2$!)

Definition 11.2. A smooth G-representation $(V, \pi) \in \mathrm{Rep}(G)$ is called *compact* if for all $v \in V \smallsetminus \{0\}$ and all compact open subgroups $K \subseteq G$, the function

$$\begin{aligned} f_{K,v}\colon G &\longrightarrow V, \\ g &\longmapsto \pi(e_K)\pi(g^{-1})v \end{aligned}$$

has compact support (hence lies in $C_c^\infty(G, V)$). Here, $e_K = \mathrm{vol}(K)^{-1}\mathbf{1}_K \in \mathcal{H}(G)$ is the idempotent from Proposition 7.4. By Lemma 7.8, we may view $\pi(e_K)$ as the projection $V \twoheadrightarrow V^K$ along $V(K)$.

Remark. If (V,π) is compact, then any subrepresentation and every quotient of V is compact. Indeed, let $W \subseteq V$ be a G-invariant subspace. It is trivial to see that (W,π) is compact. Since $f_{K,v+W}(g) = f_{K,v}(g) + W$ in V/W, for all $g \in G$, it follows that $(V/W,\pi)$ is compact.

Although the functions $f_{K,v}$ are nice to work with, it is in general not easy to check whether $f_{K,v}$ has compact support. We will next prove a necessary and sufficient criterion to verify when a representation is compact.

Definition 11.3. Let $(V,\pi) \in \mathrm{Rep}(G)$. For all $v \in V \smallsetminus \{0\}$ and $\xi \in \widetilde{V} \smallsetminus \{0\}$ we call the function

$$\begin{aligned} m_{\xi,v} \colon G &\longrightarrow \mathbb{C}, \\ g &\longmapsto \big\langle \xi, \pi(g^{-1})v \big\rangle \end{aligned}$$

a *matrix coefficient* of (V,π).

Theorem 11.4. *A smooth G-representation is compact if and only if all matrix coefficients have compact support.*

Proof. Let $(V,\pi) \in \mathrm{Rep}(G)$. Let $K \subseteq G$ be a compact open subgroup, and let $v \in V$, $\xi \in \widetilde{V}^K$, both non-zero. The functions $f_{K,v}$ and $m_{\xi,v}$ are constant on the cosets gK, hence they have compact support if and only if the image of their support in G/K is finite. Also note that, since $\xi_{|V(K)} = 0$ and $\pi(e_K)$ is the projection onto V^K, we have $\xi \circ \pi(e_K) = \xi$, and hence

$$\xi\big(f_{K,v}(g)\big) = \xi\big(\pi(e_K)\pi(g^{-1})v\big) = \xi\big(\pi(g^{-1})v\big) = m_{\xi,v}(g),$$

for all $g \in G$. Hence, we have $\operatorname{Supp} m_{\xi,v} \subseteq \operatorname{Supp} f_{K,v}$. This shows that, if (V,π) is compact, then all matrix coefficients have compact support.

Conversely, assume that all matrix coefficients have compact support. Fix a compact open subgroup $K \subseteq G$ and let $v \in V \smallsetminus \{0\}$. It suffices to find $\xi_1, \dots, \xi_n \in \widetilde{V}^K$ such that

$$\operatorname{Supp} f_{K,v} \subseteq \bigcup_{i=1}^{n} \operatorname{Supp} m_{\xi_i,v}. \tag{2.16}$$

The image of $f_{K,v}$ spans a subspace E_v of V^K. Let $\{g_i\}_{i\in I}$ be a family in G such that the $w_i \coloneqq f_{K,v}(g_i) = \pi(e_K)\pi(g_i^{-1})v$ form a $\mathbb{C}$-basis for E_v. Choose any $\xi_0 \in (V^K)^* = \widetilde{V}^K$ such that $\xi_0(w_i) = 1$ for all $i \in I$. As $\operatorname{Supp} m_{\xi_0,v}/K$ is finite and $\bigsqcup_{i\in I} g_i K \subseteq \operatorname{Supp} m_{\xi_0,v}$, it follows that I is finite, *i.e.*, E_v is finite dimensional. So let $\xi_1, \dots, \xi_n \in (V^K)^*$ whose restriction to E_v form a basis for E_v^*. For each $g \in G$ there is some i such that $m_{\xi_i,v}(g) = \xi_i(f_{K,v}(g)) \neq 0$. Thus, (2.16) is satisfied. □

Proposition 11.5. *Every finitely generated compact representation is admissible. In particular, every irreducible compact representation is admissible.*

Proof. Let (V,π) be a compact G-representation generated by, say, $v_1, \dots, v_n$. Let $K \subseteq G$ be a compact open subgroup. Note that each f_{K,v_i} has finite image, as it has compact support and is constant on the left cosets gK. Hence, the images of the $f_{K,v_1}, \dots, f_{K,v_n}$ span a finite dimensional subspace of V^K. For all $v = \sum_{i,j} a_{ij}\pi(g_{ij})v_i \in V^K$, where $a_{ij} \in \mathbb{C}$, $g_{ij} \in G$, we compute

$$v = \pi(e_K)v = \sum_{i,j} a_{ij}\pi(e_K)\pi(g_{ij})v_i = \sum_{i,j} a_{ij} f_{K,v_i}(g_{ij}^{-1}).$$

This shows that V^K is finite dimensional. Hence, V is admissible. □

Recall that G is called countable at infinity if G/K is countable for some compact open subgroup $K \subseteq G$ (Definition 7.10). The main reason we care about this notion is the following strong form of Schur's lemma:

Schur's Lemma 11.6. *Suppose G is countable at infinity. Let $(V, \pi) \in \mathrm{Rep}(G)$ be irreducible. Then*

$$\mathrm{End}_G(V) \cong \mathbb{C}.$$

In particular, if $Z(G)$ denotes the center of G, there is a smooth character $\omega_V : Z(G) \to \mathbb{C}^\times$, called the central character *of (V, π), such that $\pi(z)v = \omega_V(z)v$ for all $z \in Z(G)$ and $v \in V$.*

Proof. Fix any $v \in V$, $v \neq 0$, and let $K \subseteq G$ be a compact open subgroup with $v \in V^K$. Then $\sum_{g \in G/K} \mathbb{C}\pi(g)v$ is a non-zero G-invariant subspace of V of countable dimension. As V is irreducible, it follows that $\dim_{\mathbb{C}} V$ is countable. Moreover, the map $\mathrm{End}_G(V) \to V$, $\varphi \mapsto \varphi(v)$ is injective (since v generates V as a G-representation), and hence $\mathrm{End}_G(V)$ has countable dimension over $\mathbb{C}$. By the general version of Schur's Lemma 10.6, $\mathrm{End}_G(V)$ is a division algebra over $\mathbb{C}$.

Let $\varphi \in \mathrm{End}_G(V)$ be non-zero. Then φ is not nilpotent, and hence by Lemma 7.12(a), there exists $a \in \mathbb{C}^\times$ such that $\varphi - a\,\mathrm{id}_V$ is not left invertible. As $\mathrm{End}_G(V)$ is a division algebra, we deduce $\varphi - a\,\mathrm{id}_V = 0$.

For the existence of the central character, note that for each $z \in Z(G)$ the endomorphism $\pi(z)$ lies in $\mathrm{End}_G(V) \cong \mathbb{C}$. Hence, there exists a unique $\omega_V(z) \in \mathbb{C}^\times$ with $\pi(z) = \omega_V(z)\,\mathrm{id}_V$. One easily checks that ω_V is a smooth character. □

The main goal for this section is the following theorem:

Theorem 11.7. *Suppose G is unimodular and countable at infinity. Let $(W, \tau) \in \mathrm{Rep}(G)$ be an irreducible compact representation. Each $(V, \pi) \in \mathrm{Rep}(G)$ admits a G-equivariant decomposition*

$$V = V(\tau) \oplus V(\tau)^\perp,$$

where $V(\tau)$ is the τ-isotypic component of V, and (W, τ) does not occur as a subquotient of $V(\tau)^\perp$.

The proof needs some preparation and will be deferred to the end of the section. We assume that G is unimodular and countable at infinity.

Given any $(V, \pi) \in \mathrm{Rep}(G)$, consider the action of $G \times G$ on $\mathrm{End}_{\mathbb{C}}(V)$ given by

$$\big((g, g') \cdot \phi\big)(v) = \pi(g)\phi\big(\pi(g'^{-1})v\big), \qquad \text{for all } g, g' \in G,\, v \in V.$$

We denote $\mathrm{End}^\infty(V) \subseteq \mathrm{End}_{\mathbb{C}}(V)$ the largest smooth $G \times G$-invariant subspace.

Fix an irreducible compact G-representation (W, τ). We let $G \times G$ act on $W \otimes_{\mathbb{C}} \widetilde{W}$ by $(g, g') \cdot (w \otimes \xi) \coloneqq \tau(g)w \otimes \widetilde{\tau}(g')\xi$.

Lemma 11.8. *The map*

$$\begin{aligned} A \colon W \otimes_{\mathbb{C}} \widetilde{W} &\xrightarrow{\cong} \mathrm{End}^\infty(W), \\ w \otimes \xi &\longmapsto \big[w' \mapsto \xi(w')w\big] \end{aligned} \tag{2.17}$$

is a $G \times G$-equivariant isomorphism.

Proof. For all $g, g' \in G$, $\xi \in \widetilde{W}$ and $w, w' \in W$, we compute

$$\begin{aligned} A\big((g,g')\cdot(w\otimes\xi)\big)(w') &= A\big(\tau(g)w\otimes\widetilde{\tau}(g')\xi\big)(w') = \big(\widetilde{\tau}(g')\xi\big)(w')\cdot\tau(g)w \\ &= \xi\big(\tau(g'^{-1})w'\big)\cdot\tau(g)w = \tau(g)\big(\xi(\tau(g'^{-1})w')w\big) \\ &= \tau(g)A\big(w\otimes\xi\big)\big(\tau(g'^{-1})w'\big) \\ &= \big[(g,g')\cdot A(w\otimes\xi)\big](w'). \end{aligned}$$

This shows that A is $G\times G$-equivariant. It suffices to show that the induced map

$$A^K\colon \big(W\otimes_{\mathbb{C}}\widetilde{W}\big)^{K\times K}\longrightarrow \operatorname{End}_{\mathbb{C}}(W)^{K\times K}$$

is a $\mathbb{C}$-linear isomorphism for all compact open subgroups $K\subseteq G$. Observe that

$$\big(W\otimes_{\mathbb{C}}\widetilde{W}\big)^{K\times K} = W^K\otimes_{\mathbb{C}}\widetilde{W}^K = W^K\otimes_{\mathbb{C}}(W^K)^*,$$

cf. (2.15). Let now $\varphi\in\operatorname{End}_{\mathbb{C}}(W)^{K\times K}$ so that $\tau(k)\varphi(w)=\varphi(w)$ and $\varphi\big(\tau(k)w\big)=\varphi(w)$ for all $k\in K$ and $w\in W$. The first condition means $\varphi(W)\subseteq W^K$. The second condition means $\varphi_{|W(K)}=0$. Since $W=W^K\oplus W(K)$ by Lemma 7.8, we conclude that

$$\operatorname{End}_{\mathbb{C}}(W)^{K\times K}=\operatorname{End}_{\mathbb{C}}(W^K).$$

Under these identifications, the map A^K becomes

$$\begin{aligned} W^K\otimes_{\mathbb{C}}(W^K)^* &\xrightarrow{\cong}\operatorname{End}_{\mathbb{C}}(W^K), \\ w\otimes\xi &\longmapsto \big[w'\mapsto\xi(w')w\big], \end{aligned}$$

which is an isomorphism (W is admissible by Proposition 11.5, hence W^K is finite dimensional). □

Lemma 11.9. *Let* $(V,\pi)\in\operatorname{Rep}(G)$. *The maps*

$$\begin{aligned} m\colon W\otimes_{\mathbb{C}}\widetilde{W}&\longrightarrow\mathcal{H}(G), \\ w\otimes\xi&\longmapsto m_{\xi,w} \end{aligned}\qquad\text{and}\qquad \begin{aligned} \pi\colon\ \mathcal{H}(G)&\longrightarrow\operatorname{End}^\infty(V), \\ f&\longmapsto\pi(f) \end{aligned}$$

are well-defined and $G\times G$*-equivariant.*

Proof. Since (W,τ) is compact, Theorem 11.4 shows that the matrix coefficients $m_{\xi,w}$ lie in $\mathcal{H}(G)=C_c^\infty(G)$. If $f\in\mathcal{H}(G)$, we find a compact open subgroup $K\subseteq G$ with $e_K*f=f=f*e_K$. Lemma 7.8 shows

$$\pi(k)\pi(f)\pi(k')=\pi(k)\pi(e_K)\pi(f)\pi(e_K)\pi(k')=\pi(e_K)\pi(f)\pi(e_K)=\pi(f)$$

for all $k,k'\in K$. Hence $\pi(f)\in\operatorname{End}_{\mathbb{C}}(V)^{K\times K}\subseteq\operatorname{End}^\infty(V)$. Hence, m and π are well-defined.

Let now $g,g',x\in G$, $w\in W$, and $\xi\in\widetilde{W}$. We compute

$$\begin{aligned} m\big((g,g')(w\otimes\xi)\big)(x) &= m_{\widetilde{\tau}(g')\xi,\tau(g)w}(x)=\big\langle\widetilde{\tau}(g')\xi,\tau(x^{-1})\tau(g)w\big\rangle \\ &= \big\langle\xi,\tau(g'^{-1}x^{-1}g)w\big\rangle = m_{\xi,w}(g^{-1}xg')=\big[(g,g')m_{\xi,w}\big](x). \end{aligned}$$

Hence, m is $G\times G$-equivariant.

Similarly, for any $g, g' \in G$ and $v \in V$ we compute

$$\begin{aligned}\big[(g,g')\pi(f)\big](v) &= \pi(g)\pi(f)\big(\pi(g'^{-1})v\big) = \int_G f(x)\pi(gxg'^{-1})v \,\mathrm{d}\mu_G(x) \\ &= \delta_G(g'^{-1}) \int_G f(g^{-1}xg')\pi(x)v \,\mathrm{d}\mu_G(x) = \delta_G(g'^{-1}) \cdot \pi\big((g,g')f\big)(v) \\ &= \pi\big((g,g')f\big)(v)\end{aligned}$$

where for the last equality we have used $\delta_G(g'^{-1}) = 1$, because G is unimodular. Hence, π is $G \times G$-equivariant. $\square$

Consider now the $G \times G$-equivariant map

$$\psi \coloneqq m \circ A^{-1} \colon \operatorname{End}^\infty(W) \longrightarrow \mathcal{H}(G).$$

Proposition 11.10. *Suppose G is unimodular and countable at infinity. Let $(W,\tau) \in \operatorname{Rep}(G)$ be an irreducible compact representation.*

(a) Let $(E,\sigma) \in \operatorname{Rep}(G)$ be irreducible and not isomorphic to (W,τ). For each $f \in \operatorname{Im}(\psi) \subseteq \mathcal{H}(G)$, we have $\sigma(f) = 0$.

(b) There exists a non-zero element $d(\tau) \in \mathbb{C}^\times$ such that $\tau \circ \psi = d(\tau)^{-1} \cdot \mathrm{id}_{\operatorname{End}^\infty(W)}$.

The number $d(\tau)$ is called the formal degree *of (W,τ) (it depends on μ_G).*

Proof. We prove (a). Let $v \in E$, $v \neq 0$. By the definition of ψ, we have to show that the map

$$\begin{aligned} W \otimes_{\mathbb{C}} \widetilde{W} &\longrightarrow E, \\ w \otimes \xi &\longmapsto \sigma\big(m_{\xi,w}\big)v \end{aligned} \tag{2.18}$$

vanishes. Letting G act on the first factor of $W \otimes_{\mathbb{C}} \widetilde{W}$, we see that (2.18) is G-equivariant by Lemma 11.9. Now, $W \otimes_{\mathbb{C}} \widetilde{W}$ is τ-isotypic. As (E,σ) is not isomorphic to (W,τ), we deduce that (2.18) is the zero map.

We prove (b). By Proposition 11.5 and Exercise 10.5 it follows that $(\widetilde{W},\widetilde{\tau})$ is irreducible. Lemma 11.8 and Proposition 10.8 show that $\operatorname{End}^\infty(W) \cong W \otimes_{\mathbb{C}} \widetilde{W}$ is an irreducible $G \times G$-representation. Now, it follows from Schur's Lemma 11.6 that $\tau \circ \psi = a \cdot \mathrm{id}_{\operatorname{End}^\infty(W)}$ for some scalar $a \in \mathbb{C}$. We have to show $a \neq 0$. Let $f \in \operatorname{Im}(\psi) \subseteq \mathcal{H}(G)$. The Separation Lemma 7.11 provides an irreducible representation $(E,\sigma) \in \operatorname{Rep}(G)$ with $\sigma(f) \neq 0$. From (a) we deduce $(E,\sigma) \cong (W,\tau)$, and hence $\tau \circ \psi \neq 0$. $\square$

Exercise. Suppose G is profinite, and let $(W,\tau) \in \operatorname{Rep}(G)$ be an irreducible representation. Show that $d(\tau) = \frac{\dim W}{\operatorname{vol}(G;\mu_G)}$.

(Hint: First, show $\langle \eta, w\rangle \cdot \langle \xi, v\rangle = d(\tau) \int_G \langle \widetilde{\tau}(x)\xi, w\rangle \cdot \langle \eta, \tau(x)v\rangle \,\mathrm{d}\mu_G(x)$ for all $v, w \in W$ and $\xi, \eta \in \widetilde{W} = W^*$. For a $\mathbb{C}$-basis $w_1, \ldots, w_d \in W$ with dual basis $\eta_1, \ldots, \eta_d \in W^*$, compute $\sum_{i,j=1}^d \langle \eta_i, w_i\rangle\langle \eta_j, w_j\rangle$ in two ways.)

Proposition 11.11. *Suppose G is unimodular and countable at infinity. Fix an irreducible compact representation $(W,\tau) \in \operatorname{Rep}(G)$ and let $K \subseteq G$ be a compact open subgroup. Define*

$$e_{K,\tau} \coloneqq d(\tau) \cdot (\psi \circ \tau)(e_K) \in \mathcal{H}(G).$$

(a) *$e_{K,\tau}$ is the unique element with $\tau(e_{K,\tau}) = \tau(e_K)$ and $\sigma(e_{K,\tau}) = 0$ for each irreducible smooth representation $(E,\sigma) \not\cong (W,\tau)$.*

(b) *For each open subgroup $H \subseteq K$ one has*

$$\begin{aligned} e_{H,\tau} * e_{K,\tau} &= e_{K,\tau} = e_{K,\tau} * e_{H,\tau}, \\ e_{H,\tau} * e_K &= e_{K,\tau} = e_K * e_{H,\tau}, \\ e_{K,\tau} * e_H &= e_{K,\tau} = e_H * e_{K,\tau}. \end{aligned}$$

In particular, $e_{K,\tau}$ is an idempotent.

(c) *For each $(V,\pi) \in \mathrm{Rep}(G)$ and $g \in G$ one has $\pi(g)\pi(e_{K,\tau}) = \pi(e_{gKg^{-1},\tau})\pi(g)$.*

Proof. The uniqueness follows immediately from the Separation Lemma 7.11. Proposition 11.10 shows that $e_{K,\tau}$ has the required properties, whence (a).

The identities in (b) follow from the Separation Lemma 7.11, Proposition 7.4, and from (a). For example, we have $\tau(e_{H,\tau} * e_K) = \tau(e_{H,\tau})\tau(e_K) = \tau(e_H)\tau(e_K) = \tau(e_H * e_K) = \tau(e_K) = \tau(e_{K,\tau})$ and $\sigma(e_{H,\tau} * e_K) = \sigma(e_{H,\tau})\sigma(e_K) = 0 = \sigma(e_{K,\tau})$ for all irreducible smooth $(E,\sigma) \not\cong (W,\tau)$.

Let us prove (c). Let $H \subseteq K$ be an open subgroup and fix $g \in G$. Using Lemma 7.3 and the fact that $\psi \circ \tau$ is $G \times G$-equivariant, we compute

$$\begin{aligned} \big(\lambda(g)e_{K,\tau}\big) * e_H &= \big(\rho(g^{-1})\rho(g)\lambda(g)e_{K,\tau}\big) * e_H = \delta_G(g^{-1}) \cdot \big(\rho(g)\lambda(g)e_{K,\tau}\big) * \big(\lambda(g)e_H\big) \\ &= \delta_G(g^{-1}) \cdot e_{gKg^{-1},\tau} * e_{gH} = e_{gKg^{-1},\tau} * e_{gH}. \end{aligned}$$

Now, let $(V,\pi) \in \mathrm{Rep}(G)$ and $v \in V$. Choose an open subgroup $H \subseteq K$ with $v \in V^H$. Then

$$\pi(g)\pi(e_{K,\tau})v = \pi\big(\lambda(g)e_{K,\tau} * e_H\big)v = \pi\big(e_{gKg^{-1},\tau} * e_{gH}\big)v = \pi(e_{gKg^{-1},\tau})\pi(g)v. \qquad \square$$

We fix a smooth representation (V,π) of G. For each $v \in V^K$, we put

$$\pi(e_\tau)v := \pi(e_{K,\tau})v.$$

Since $\pi(e_{H,\tau})v = \pi(e_{K,\tau})v$ for all compact open subgroups $H \subseteq K$ (Proposition 11.11(b)), this gives a well-defined $\mathbb{C}$-linear map

$$\pi(e_\tau)\colon V \to V.$$

We now prove Theorem 11.7 in the following stronger form:

Theorem 11.12. *Suppose G is unimodular and countable at infinity. Let $(W,\tau) \in \mathrm{Rep}(G)$ be an irreducible compact representation. Let $(V,\pi) \in \mathrm{Rep}(G)$.*

(a) *The map $\pi(e_\tau)$ is a G-equivariant projection.*

(b) *Let $(V',\pi') \in \mathrm{Rep}(G)$ and let $\alpha \in \mathrm{Hom}_G(V,V')$. Then $\alpha \circ \pi(e_\tau) = \pi'(e_\tau) \circ \alpha$.*

(c) *One has a decomposition*

$$V = \mathrm{Im}\,\pi(e_\tau) \oplus \mathrm{Ker}\,\pi(e_\tau)$$

as G-representations, where $\mathrm{Im}\,\pi(e_\tau)$ is τ-isotypic, and (W,τ) does not occur as a subquotient of $\mathrm{Ker}(e_\tau)$.

Proof. Parts (a) and (b) follow from Proposition 11.11(b)/(c); just note that, if $v \in V^K$, $g \in G$ and $\alpha \in \mathrm{Hom}_G(V, V')$, then $\pi(g)v \in V^{gKg^{-1}}$ and $\alpha(v) \in (V')^K$.

Let us prove (c). By (a), we have the decomposition $V = \mathrm{Im}\,\pi(e_\tau) \oplus \mathrm{Ker}\,\pi(e_\tau)$. We show that $\mathrm{Im}\,\pi(e_\tau)$ is τ-isotypic. Let $v \in V$ be arbitrary and fix a compact open subgroup $K \subseteq G$ with $v \in V^K$. Consider the diagram

$$\begin{array}{ccccc} f * e_{K,\tau} & \in & \mathcal{H}(G) * e_{K,\tau} & \subseteq & \mathrm{Im}\big(m\colon W \otimes_{\mathbb{C}} \widetilde{W} \to \mathcal{H}(G)\big) \\ \big\downarrow & & \big\downarrow{\scriptstyle \phi_v} & & \\ \pi(f * e_{K,\tau})v & \in & V & & \end{array}$$

Note that $e_{K,\tau} = d(\tau) \cdot (m \circ A^{-1} \circ \tau)(e_K) \in \mathrm{Im}(m)$. Letting G act only on the first factor of $W \otimes_{\mathbb{C}} \widetilde{W}$, it is clear that $W \otimes_{\mathbb{C}} \widetilde{W}$ is τ-isotypic. As m is G-equivariant, we deduce that $\mathrm{Im}(m)$ and hence also $\mathcal{H}(G) * e_{K,\tau}$ is τ-isotypic. As ϕ_v is G-equivariant, we deduce that $\mathrm{Im}(\phi_v)$ is τ-isotypic. But then, also $\mathrm{Im}(\pi(e_\tau)) = \sum_{v \in V} \mathrm{Im}(\phi_v)$ is τ-isotypic.

We now prove that (W, τ) does not occur as a subquotient of $\mathrm{Ker}\,\pi(e_\tau)$. Since $\tau(e_\tau) = \mathrm{id}_W$, it suffices to show that $\pi(e_\tau)$ annihilates any subquotient of $\mathrm{Ker}\,\pi(e_\tau)$. So let (V'', π'') be a subquotient of $\mathrm{Ker}\,\pi(e_\tau)$. Then there exists a G-invariant subspace $V' \subseteq \mathrm{Ker}\,\pi(e_\tau)$ and surjective G-equivariant map $\alpha\colon V' \twoheadrightarrow V''$. Then $\pi''(e_\tau)V'' = \pi''(e_\tau)\alpha(V') = \alpha\big(\pi(e_\tau)V'\big) = \{0\}$. □

Corollary 11.13. *Every compact representation* $(V, \pi) \in \mathrm{Rep}(G)$ *is semisimple.*

Proof. Let $V' \subseteq V$ be the sum of all irreducible subrepresentations. By Proposition 8.1 it suffices to prove $V' = V$. Assume for a contradiction that $V/V' \neq 0$. Let (W, τ) be an irreducible subquotient of $(V/V', \pi'')$. Then $\pi''(e_\tau)(V/V') \neq \{0\}$ by Theorem 11.12 and hence we have $\pi(e_\tau)V \not\subseteq V'$. But this contradicts the fact that $\pi(e_\tau)V$ is τ-isotypic and in particular the sum of its irreducible subrepresentations (each of which is isomorphic to (W, τ)). □

Corollary 11.14 (Obsolete)**.** *Every irreducible compact representation* (W, τ) *is projective and injective in* $\mathrm{Rep}(G)$.

Proof. Note that the functor $\mathrm{Rep}(G) \to \mathrm{Rep}(G)$, $(V, \pi) \mapsto \pi(e_\tau)V$ is exact. Since (W, τ) is projective in the category of compact representations by Corollary 11.13, it follows that the functor

$$\begin{aligned} \mathrm{Rep}(G) &\longrightarrow \mathrm{Vect}_{\mathbb{C}}, \\ (V, \pi) &\longmapsto \mathrm{Hom}_G\big(W, \pi(e_\tau)V\big) = \mathrm{Hom}_G(W, V) \end{aligned}$$

is exact. Hence (W, τ) is projective in $\mathrm{Rep}(G)$. A similar argument shows that (W, τ) is injective in $\mathrm{Rep}(G)$. □

Chapter 3

Smooth Representations of p-Adic Groups

Throughout this chapter, we fix a local field F with valuation ring o_F, maximal ideal $\mathfrak{m}_F$, residue field κ_F, and uniformizer ϖ. Recall the associated discrete valuation

$$\mathrm{val}_F \colon F \twoheadrightarrow \mathbb{Z} \cup \{\infty\},$$

which is given by $\mathrm{val}_F(x) = \sup\{n \in \mathbb{Z} \mid x \in \varpi^n o_F\}$.

§12. Decompositions of $\mathrm{GL}_n(F)$

Recall the group $\mathrm{GL}_n(F)$ of invertible $n \times n$-matrices. We have seen in Example 4.4 that $\mathrm{GL}_n(F)$ is locally profinite, that $\mathrm{GL}_n(o_F) \subseteq \mathrm{GL}_n(F)$ is a compact open subgroup, and that the congruence subgroups

$$K_m := 1 + \varpi^m \operatorname{Mat}_{n,n}(o_F), \qquad \text{for } m \geqslant 1,$$

form a system of fundamental open subgroups of $\mathrm{GL}_n(F)$, which are normal in $\mathrm{GL}_n(o_F)$. In this section, we will study in detail the structure of $\mathrm{GL}_n(F)$. We start with describing the maximal compact subgroups of $\mathrm{GL}_n(F)$.

Definition 12.1. A *lattice* in F^n is a finitely generated o_F-submodule $\mathcal{L} \subseteq F^n$ which generates F^n as an F-vector space.

Lemma 12.2. *Let $\mathcal{L} \subseteq F^n$ be a lattice. Then there exists an F-basis $x_1, \ldots, x_n \in F$ such that $\mathcal{L} = \bigoplus_{i=1}^n o_F.x_i$. (In particular, $\mathcal{L}$ is a free o_F-module of rank n.)*

Proof. Let $y_1, \ldots, y_m$ be a minimal generating system of $\mathcal{L}$ as an o_F-module. We claim this is an F-basis of F^n. Obviously, it generates F^n as a vector space. It is also linearly independent: Let $\sum_{i=1}^m a_i y_i = 0$ with $a_i \in F$, not all of them zero. Fix j with $\mathrm{val}_F(a_j) \leqslant \mathrm{val}_F(a_i)$ for all i. Then $\mathrm{val}_F(a_j^{-1}a_i) \geqslant 0$ for all i, and hence $y_j = -\sum_{i \neq j} a_j^{-1} a_i y_i$ is an o_F-linear combination, which contradicts the fact that $y_1, \ldots, y_m$ is a minimal set of generators of $\mathcal{L}$. □

Proposition 12.3. $\mathrm{GL}_n(o_F)$ *is a maximal compact (open) subgroup of* $\mathrm{GL}_n(F)$*. Every compact subgroup of* $\mathrm{GL}_n(F)$ *is conjugate to a subgroup of* $\mathrm{GL}_n(o_F)$*.*

Proof. We first show that $\mathrm{GL}_n(o_F)$ is maximal. Let $H \subseteq \mathrm{GL}_n(F)$ be a subgroup strictly containing $\mathrm{GL}_n(o_F)$. Take $A = (a_{ij})_{i,j} \in H \smallsetminus \mathrm{GL}_n(o_F)$. Replacing A with A^{-1} if necessary, we find i_0, j_0 such that $\mathrm{val}_F(a_{i_0 j_0})$ is negative and minimal among all $\mathrm{val}_F(a_{ij})$. Multiplying A with suitable matrices in $\mathrm{GL}_n(o_F)$, we may assume that $i_0 = 1 = j_0$ and $a_{1i} = 0$ for all $i > 1$. Then a_{11}^r is the $(1,1)$-entry of $A^r \in H$. It follows that $H = \bigcup_{r \geqslant 0} \varpi^{-r} \mathrm{Mat}_{n,n}(o_F) \cap H$ does not admit a finite subcover. Hence, H is not compact.

Let $H \subseteq \mathrm{GL}_n(F)$ be a compact subgroup. Let $e_1, \dots, e_n$ be the standard basis of F^n and put $\mathcal{L} = \bigoplus_{i=1}^n o_F.e_i$. Denote $\mathcal{L}_H$ the smallest H-invariant o_F-module containing $\mathcal{L}$. Then $\mathcal{L}_H$ is generated as an o_F-module by the image C of the continuous map

$$\begin{aligned} \{1,2,\dots,n\} \times H &\longrightarrow F^n, \\ (i,h) &\longmapsto h(e_i). \end{aligned}$$

As $\{1,2,\dots,n\} \times H$ is compact, it follows that C is compact. As $F^n = \bigcup_{m \in \mathbb{Z}_{\geqslant 0}} \varpi^{-m}\mathcal{L}$ is an open covering, there exists $m \in \mathbb{Z}$ such that $C \subseteq \varpi^{-m}\mathcal{L}$. We deduce that $\mathcal{L}_H \subseteq \varpi^{-m}\mathcal{L}$ is finitely generated, because o_F is Noetherian. We have shown that $\mathcal{L}_H$ is an H-invariant lattice. By Lemma 12.2, there exists an F-basis $x_1, \dots, x_n$ in F^n with $\mathcal{L}_H = \bigoplus_{i=1}^n o_F.x_i$. Let $g\colon F^n \to F^n$ be the F-linear automorphism such that $g(x_i) = e_i$ viewed as an $n \times n$-matrix with respect to $e_1, \dots, e_n$. Then gHg^{-1} stabilizes $\mathcal{L} = \bigoplus_{i=1}^n o_F.e_i$ and is therefore contained in $\mathrm{GL}_n(o_F)$. □

We now put $G \coloneqq \mathrm{GL}_n(F)$ and $K \coloneqq \mathrm{GL}_n(o_F)$.

Notation.
- Denote Σ_n the *symmetric group* on n elements. For each $\sigma \in \Sigma_n$ we denote

$$w_\sigma \coloneqq \big(\delta_{i,\sigma(j)}\big)_{i,j} \in K$$

  the *permutation matrix* associated with σ; it is characterized by $w_\sigma e_i = e_{\sigma(i)}$. Here, δ_{ij} is the *Kronecker-delta*, defined by $\delta_{ij} \coloneqq 1$ if $i = j$ and $\delta_{ij} \coloneqq 0$ if $i \neq j$.
- Put

$$\begin{aligned} \Lambda &\coloneqq \{\mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n}) \,|\, m_1, \dots, m_n \in \mathbb{Z}\} \cong \mathbb{Z}^n, \\ \Lambda^+(G) \coloneqq \Lambda^+ &\coloneqq \{\mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n}) \in \Lambda \,|\, m_1 \geqslant m_2 \geqslant \dots \geqslant m_n\}. \end{aligned}$$

Theorem 12.4 (Cartan decomposition)**.** *One has a disjoint decomposition*

$$G = \bigsqcup_{\lambda \in \Lambda^+} K\lambda K,$$

that is, Λ^+ is a complete set of representatives of the double coset space $K\backslash G/K$.

Proof. Let $A = (a_{ij})_{i,j} \in G$. Fix i_0, j_0 such that $\mathrm{val}_F(a_{i_0 j_0}) = \min\{\mathrm{val}_F(a_{ij}) \,|\, 1 \leqslant i,j \leqslant n\}$. Replacing A by $w_{(n\, i_0)} A w_{(n\, j_0)}$ if necessary, we may assume that $i_0 = j_0 = n$. Write $a_{nn} = x\varpi^{m_n}$, for $x \in o_F^\times$. Then $B \coloneqq \mathrm{diag}(1, \dots, 1, x^{-1}) \in K$ and hence, replacing A with AB if necessary, we may assume $a_{nn} = \varpi^{m_n}$. Now note that

$$\left(\begin{array}{c|c} E_{n-1} & \begin{matrix} -\frac{a_{1n}}{a_{nn}} \\ \vdots \\ -\frac{a_{n-1,n}}{a_{nn}} \end{matrix} \\ \hline 0 \cdots 0 & 1 \end{array}\right) \left(\begin{array}{c|c} * & \begin{matrix} a_{1n} \\ \vdots \\ a_{n-1,n} \end{matrix} \\ \hline a_{n1} \cdots a_{n,n-1} & a_{nn} \end{array}\right) \left(\begin{array}{c|c} E_{n-1} & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline -\frac{a_{n1}}{a_{nn}} \cdots -\frac{a_{n,n-1}}{a_{nn}} & 1 \end{array}\right) = \left(\begin{array}{c|c} A' & \begin{matrix} 0 \\ \vdots \\ 0 \end{matrix} \\ \hline 0 \cdots 0 & \varpi^{m_n} \end{array}\right)$$

lies in KAK and every entry of A' has valuation $\geqslant m_n$. By induction, we see that KAK contains a matrix of the form $\mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n})$ with $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$.

It remains to see that the union in the assertion is disjoint. Let $m_1, \dots, m_n, m_1', \dots, m_n' \in \mathbb{Z}$ such that

$$K \begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix} K = K \begin{pmatrix} \varpi^{m_1'} & & \\ & \ddots & \\ & & \varpi^{m_n'} \end{pmatrix} K$$

It suffices to find $\sigma \in \Sigma_n$ with $m_i = m'_{\sigma(i)}$ for all $1 \leqslant i \leqslant n$. Let $A = (a_{ij})_{i,j} \in K$ such that

$$X \coloneqq \begin{pmatrix} \varpi^{m_1} & & \\ & \ddots & \\ & & \varpi^{m_n} \end{pmatrix} A \begin{pmatrix} \varpi^{-m_1'} & & \\ & \ddots & \\ & & \varpi^{-m_n'} \end{pmatrix} \in K.$$

We have $0 = \mathrm{val}_F(\det(X)) = \sum_{i=1}^n m_i + \mathrm{val}_F(\det(A)) - \sum_{i=1}^n m_i' = \sum_{i=1}^n m_i - \sum_{i=1}^n m_i'$. Recall the Leibniz formula

$$\det(A) = \sum_{\sigma \in \Sigma_n} \mathrm{sgn}(\sigma) \cdot a_{1\sigma(1)} \cdots a_{n\sigma(n)} \in o_F^\times.$$

As $A \in K$, we find $\sigma \in \Sigma_n$ with $a_{i\sigma(i)} \in o_F^\times$ for all i. Since $X = (x_{ij})_{i,j} \in K$, we have $\varpi^{m_i - m'_{\sigma(i)}} a_{i\sigma(i)} = x_{i\sigma(i)} \in o_F$, which shows $m_i - m'_{\sigma(i)} \geqslant 0$, for all i. Now, $\sum_{i=1}^n (m_i - m'_{\sigma(i)}) = \sum_{i=1}^n m_i - \sum_{i=1}^n m'_{\sigma(i)} = 0$, so we conclude $m_i = m'_{\sigma(i)}$ for all i. This finishes the proof. □

Exercise (Elementary divisor theorem for o_F). Let $\mathcal{L}_1, \mathcal{L}_2$ be two o_F-lattices in F^n. Show that there is an o_F-basis $e_1, \dots, e_n$ of $\mathcal{L}_1$ and uniquely determined integers $m_1 \geqslant m_2 \geqslant \cdots \geqslant m_n$ such that $\pi^{m_1} e_1, \dots, \pi^{m_n} e_n$ is an o_F-basis of $\mathcal{L}_2$.

Corollary 12.5. *Let $H \subseteq G = \mathrm{GL}_n(F)$ be a closed subgroup. Then H is countable at infinity. Moreover, the center $Z(H)$ acts through a character on every irreducible smooth H-representation.*

Proof. Since $H/H \cap K \subseteq G/K$, it suffices to show that G is countable at infinity. By the Cartan decomposition 12.4, we have $G = \bigsqcup_{\lambda \in \Lambda^+} K\lambda K \subseteq \bigcup_{\lambda \in \Lambda^+} \bigcup_{k \in K/(\lambda K \lambda^{-1} \cap K)} k\lambda K$. As Λ^+ is countable and each $K/(\lambda K \lambda^{-1} \cap K)$ is finite (since K is compact), it follows that G/K is countable. The last assertion is now a consequence of Schur's Lemma 11.6. □

We now consider the subgroups

$$B \coloneqq \begin{pmatrix} * & \cdots & * \\ & \ddots & \vdots \\ 0 & & * \end{pmatrix}, \quad T \coloneqq \begin{pmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{pmatrix}, \quad U \coloneqq \begin{pmatrix} 1 & * & \cdots & * \\ & \ddots & \ddots & \vdots \\ & & \ddots & * \\ 0 & & & 1 \end{pmatrix}$$

of $G = \mathrm{GL}_n(F)$. Note that U is a normal subgroup of B, and $B = TU = UT$. Put

$$\mathcal{W} \coloneqq \{w_\sigma \mid \sigma \in \Sigma_n\} \cong \Sigma_n.$$

Definition 12.6. We call

- B the *standard Borel subgroup* of G;
- T the *standard maximal torus* of G;

– U the *unipotent radical* of B and

– $\mathcal{W}$ the *Weyl group* of G (with respect to T).

Theorem 12.7 (Iwasawa decomposition)**.** *We have $G = KB = BK$. In particular, G/B is compact.*

Proof. Let $A = (a_{ij})_{i,j} \in G$. We need to find $k \in K$ such that $kA \in B$. Since $\mathcal{W} \subseteq K$, we find $\sigma \in \Sigma_n$ such that $\operatorname{val}_F(a_{\sigma(1),1}) \leqslant \operatorname{val}_F(a_{i1})$ for all i. Replacing A with $w_{\sigma^{-1}}A$, we may assume $\operatorname{val}_F(a_{11}) \leqslant \operatorname{val}_F(a_{i1})$ for all i. As before, we have

$$\left(\begin{array}{c|c} 1 & 0 \cdots 0 \\ \hline -\frac{a_{21}}{a_{11}} & \\ \vdots & E_{n-1} \\ -\frac{a_{n1}}{a_{11}} & \end{array}\right) \cdot A = \left(\begin{array}{c|c} a_{11} & a_{12} \cdots a_{1n} \\ \hline 0 & \\ \vdots & A' \\ 0 & \end{array}\right) \in KA.$$

By induction on n, we find $k \in K$ with $kA \in B$, which proves $G = KB$. Now, $G = G^{-1} = (KB)^{-1} = B^{-1}K^{-1} = BK$.

Finally, note that we have a continuous surjection $K \twoheadrightarrow G/B$. As K is compact, so is G/B. □

Lemma 12.8. *Let $N_G(T) \coloneqq \left\{g \in G \,\middle|\, gTg^{-1} = T\right\}$ be the* normalizer *of T in G. Then $N_G(T)/T \cong \mathcal{W}$.*

Proof. We need to show $N_G(T) = T\mathcal{W} = \mathcal{W}T$. It is clear that $\mathcal{W}$ normalizes T. Conversely, let $a = (a_{ij})_{i,j} \in N_G(T)$. Assume for a contradiction that there exists $1 \leqslant i \leqslant n$ and $j_1 \neq j_2$ such that $a_{i,j_1} \neq 0 \neq a_{i,j_2}$. Choose $t = \operatorname{diag}(t_1, \dots, t_n) \in T$ with $t_{j_1} \neq t_{j_2}$. By assumption, there exists $t' = \operatorname{diag}(t'_1, \dots, t'_n) \in T$ with $at = t'a$. We compute

$$\begin{aligned} a_{i,j_1}t_{j_1} = (at)_{i,j_1} = (t'a)_{i,j_1} = t'_i a_{i,j_1} &= \frac{a_{i,j_1}}{a_{i,j_2}} \cdot t'_i a_{i,j_2} \\ &= \frac{a_{i,j_1}}{a_{i,j_2}} \cdot a_{i,j_2}t_{j_2} = a_{i,j_1}t_{j_2}. \end{aligned}$$

Since $a_{i,j_1} \neq 0$, this means $t_{j_1} = t_{j_2}$ which contradicts $t_{j_1} \neq t_{j_2}$. Hence $a \in \mathcal{W}T$. □

Theorem 12.9 (Bruhat decomposition)**.** *One has a disjoint decomposition*

$$G = \bigsqcup_{w \in \mathcal{W}} BwB.$$

Moreover, $BwB = UwB = BwU$ for all $w \in \mathcal{W}$.

Proof. Note that U is generated by the *elementary matrices* $e_{ij}(x)$ for $1 \leqslant i < j \leqslant n$ and $x \in F$, given by

$$\big(e_{ij}(x)\big)_{r,s} \coloneqq \begin{cases} 1, & \text{if } r = s, \\ x, & \text{if } (r,s) = (i,j), \\ 0, & \text{otherwise.} \end{cases}$$

Now verify that any element of G can be transformed into an element of $T\mathcal{W}$ by multiplying with elementary matrices from the left and right. Since each $w \in \mathcal{W}$ normalizes T, and since $B = TU = UT$, we have $BwB = UwB = BwU$.

It remains to prove $Bw_\sigma B \neq Bw_\tau B$ whenever $\sigma \neq \tau$ in Σ_n. Assume otherwise, and let $u \in U$ such that $w_\sigma u w_\tau^{-1} \in B$. Fix i such that $\sigma(i) > \tau(i)$. The $(\sigma(i), \tau(i))$-th entry of $w_\sigma u w_\tau^{-1}$ then equals $u_{ii} = 1$, which contradicts the fact that $w_\sigma u w_\tau^{-1} \in B$. □

Definition 12.10. A *partition* of $n \in \mathbb{Z}_{\geqslant 1}$ is a tuple $\underline{n} = (n_1, n_2, \ldots, n_r)$, where $n_1, \ldots, n_r \in \mathbb{Z}_{\geqslant 1}$ such that $n_1 + \cdots + n_r = n$. If $\underline{n}' = (n_1', \ldots, n_s')$ is another partition of n, we write $\underline{n} \leqslant \underline{n}'$ if there are integers $0 = r_0 < r_1 < r_2 < \cdots < r_s = r$ such that $n_i' = \sum_{j=r_{i-1}+1}^{r_i} n_j$ for $1 \leqslant i \leqslant s$. This defines a partial order on the set of all partitions of n. For example, we have

$$(1, 2, 3, 4) \leqslant (3, 3, 4) \leqslant (3, 7) \leqslant (10)$$

as partitions of 10.

Let $\underline{n} = (n_1, \ldots, n_r)$ be a partition of n. The subgroup $P_{\underline{n}}$ of G consisting of matrices of the form

$$\begin{matrix} n_1 \\ n_2 \\ \\ n_r \end{matrix}\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1r} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{r-1,r} \\ 0 & \cdots & 0 & A_{rr} \end{pmatrix},$$

where $A_{ii} \in \mathrm{GL}_{n_i}(F)$ for all $1 \leqslant i \leqslant r$, and $A_{ij} \in \mathrm{Mat}_{n_i,n_j}(F)$ for all $1 \leqslant i < j \leqslant r$, is called a *standard parabolic subgroup* of shape $\underline{n}$.

The subgroup $U_{\underline{n}}$ of $P_{\underline{n}}$ consisting of the matrices of the form

$$\begin{matrix} n_1 \\ n_2 \\ \\ n_r \end{matrix}\begin{pmatrix} E_{n_1} & A_{12} & \cdots & A_{1r} \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & A_{r-1,r} \\ 0 & \cdots & 0 & E_{n_r} \end{pmatrix},$$

where $E_{n_i} \in \mathrm{GL}_{n_i}(F)$ denotes the identity matrix, is called the *unipotent radical* of $P_{\underline{n}}$.

The subgroup $M_{\underline{n}}$ of $P_{\underline{n}}$ consisting of the block diagonal matrices

$$\begin{matrix} n_1 \\ n_2 \\ \\ n_r \end{matrix}\begin{pmatrix} A_{11} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & A_{rr} \end{pmatrix},$$

where $A_{ii} \in \mathrm{GL}_{n_i}(F)$ for all $1 \leqslant i \leqslant r$, is called the *standard Levi subgroup* of $P_{\underline{n}}$.

We denote by $\overline{P}_{\underline{n}}$ and $\overline{U}_{\underline{n}}$ the transpose of $P_{\underline{n}}$ and $U_{\underline{n}}$, respectively. We call $\overline{P}_{\underline{n}}$ the *opposite parabolic* of $P_{\underline{n}}$.

A subgroup P of G is called *parabolic* if there exists $g \in G$ such that gPg^{-1} is standard parabolic. Similarly, a subgroup M of G is called a *Levi subgroup* if there exists $g \in G$ such that gMg^{-1} is a standard Levi subgroup.

We observe the following easy facts:

– $M_{\underline{n}} \cong \mathrm{GL}_{n_1}(F) \times \cdots \times \mathrm{GL}_{n_r}(F)$.

– $U_{\underline{n}}$ is a normal subgroup in $P_{\underline{n}}$ and $P_{\underline{n}} = M_{\underline{n}} U_{\underline{n}} = U_{\underline{n}} M_{\underline{n}}$ and $M_{\underline{n}} \cap U_{\underline{n}} = \{1\}$.

– $B \subseteq P_{\underline{n}}$, $T \subseteq M_{\underline{n}}$ are subgroups, and $U_{\underline{n}} \subseteq U$ is a normal subgroup.

– More generally, if $\underline{n} \leqslant \underline{n}'$, then $P_{\underline{n}} \subseteq P_{\underline{n}'}$, $M_{\underline{n}} \subseteq M_{\underline{n}'}$ and $U_{\underline{n}} \supseteq U_{\underline{n}'}$. In this case, we have

$$P_{\underline{n}} \cap M_{\underline{n}'} = M_{\underline{n}} \cdot (U_{\underline{n}} \cap M_{\underline{n}'}).$$

We call $P_{\underline{n}} \cap M_{\underline{n}'}$ a standard parabolic subgroup of $M_{\underline{n}'}$ with standard Levi subgroup $M_{\underline{n}}$ and unipotent radical $U_{\underline{n}} \cap M_{\underline{n}'}$.

– $\overline{P}_{\underline{n}} = M_{\underline{n}}\overline{U}_{\underline{n}} = \overline{U}_{\underline{n}}M_{\underline{n}}$.

– $P_{\underline{n}} \cap \overline{P}_{\underline{n}} = M_{\underline{n}}$.

Example 12.11. (a) $P_{(1,\dots,1)} = B$, $U_{(1,\dots,1)} = U$, and $M_{(1,\dots,1)} = T$. Further, $\overline{P}_{(1,\dots,1)}$ consists of the lower triangular matrices in G.

(b) $P_{(n)} = M_{(n)} = G$ and $U_{(n)} = \{1\}$.

Exercise 12.12. (a) Let $g \in G$ such that $gTg^{-1} \subseteq B$. Show that $gTg^{-1} = bTb^{-1}$ for some $b \in B$. (Hint: Let $t := \operatorname{diag}(t_1, \dots, t_n) \in T$ such that $t_i \neq t_j$ for $i \neq j$. Show $T = Z_G(t) := \{x \in G \mid xt = tx\}$. Deduce that it suffices to find $b \in B$ with $gtg^{-1} \in bTb^{-1}$. Next, show that gtg^{-1} stabilizes the subspaces $V_i := Fe_1 + \cdots + Fe_i \subseteq F^n$, for all $1 \leqslant i \leqslant n$ (where $e_1, \dots, e_n$ denotes the standard basis of F^n). Show that there exists a permutation $\sigma \in \Sigma_n$ such that $V_i = Fge_{\sigma(1)} \oplus \cdots \oplus Fge_{\sigma(i)}$ is the eigenspace decomposition for gtg^{-1}. Deduce $gw_\sigma \in B$ and conclude.)

(b) Show that the set $\{gBg^{-1} \mid g \in G \text{ and } gBg^{-1} \supseteq T\}$ is in bijection with $\mathcal{W}$ (and in particular finite). (Hint: (a) and Lemma 12.8.)

(c) Let $M \subseteq G$ be a standard Levi subgroup. Let $\mathcal{P}(M)$ be the set of parabolic subgroups of G with Levi subgroup M. Show that $\mathcal{P}(M)$ is finite.

(d) Let $M \subseteq G$ be a standard Levi subgroup and put $\mathcal{W}(M) := N_G(M)/M$, where $N_G(M) = \{g \in G \mid gMg^{-1} = M\}$ is the normalizer of M in G. Show that the group homomorphism $N_G(M) \cap \mathcal{W} \to \mathcal{W}(M)$ is surjective (what is the kernel?). In particular, $\mathcal{W}(M)$ is finite. (Hint: Let $g \in N_G(M)$ so that $gTg^{-1} \subseteq M$. Using the strategy in (a), show that there exists $m \in M$ such that $mgT(mg)^{-1} = T$.)

We fix a partition $\underline{n} = (n_1, \dots, n_r)$ of n.

Lemma 12.13. *The multiplication map*

$$\overline{U}_{\underline{n}} \times M_{\underline{n}} \times U_{\underline{n}} \longrightarrow G$$

is injective (but not a group homomorphism).

Proof. Take $\overline{u}_1, \overline{u}_2 \in \overline{U}_{\underline{n}}$, $m_1, m_2 \in M_{\underline{n}}$, and $u_1, u_2 \in U_{\underline{n}}$ such that

$$\overline{u}_1 m_1 u_1 = \overline{u}_2 m_2 u_2.$$

Then $\overline{u}_2^{-1}\overline{u}_1 = (m_2u_2u_1^{-1}m_2^{-1}) \cdot (m_2m_1^{-1}) \in \overline{U}_{\underline{n}} \cap P_{\underline{n}} = \{1\}$. We deduce $\overline{u}_1 = \overline{u}_2$. Since $M_{\underline{n}} \cap U_{\underline{n}} = \{1\}$, we further deduce $m_1 = m_2$ and $m_2u_2u_1^{-1}m_2^{-1} = 1$. The latter is equivalent to $u_1 = u_2$. ☐

Notation 12.14. (a) If $\underline{n}' = (n_1', \dots, n_s')$ is a partition of n, we define

$$\Lambda^{++}(M_{\underline{n}'}) := \left\{\operatorname{diag}(\varpi^{m_1}E_{n_1'}, \dots, \varpi^{m_s}E_{n_s'}) \in \Lambda^+ \cap Z(M_{\underline{n}'}) \,\middle|\, m_1 > m_2 > \cdots > m_s\right\}.$$

(b) More generally, let $\underline{n}' = (n_1', \dots, n_s') \leqslant \underline{n} = (n_1, \dots, n_r)$ be partitions of n. We may identify $\underline{n}'$ with a tuple $(\underline{n}_1', \dots, \underline{n}_r')$, where each $\underline{n}_i'$ is a partition of n_i. We define

$$\Lambda^{++}(M_{\underline{n}'}, M_{\underline{n}}) = \left\{ \operatorname{diag}(\lambda_1, \dots, \lambda_r) \,\middle|\, \lambda_i \in \Lambda^{++}(M_{\underline{n}_i'}) \text{ for all } 1 \leqslant i \leqslant r \right\}.$$

For example, we have

$$\left(\begin{array}{cc|cc} \varpi^2 & & & \\ & \varpi^2 & & \\ \hline & & \varpi^3 & \\ & & & \varpi \end{array}\right) \in \Lambda^{++}\big(M_{(2,1,1)}, M_{(2,2)}\big).$$

Proposition 12.15. *Let $K_m = 1 + \varpi^m \operatorname{Mat}_{n,n}(o_F)$ be the m-th congruence subgroup, where $m \geqslant 1$. Let $\underline{n} = (n_1, \dots, n_r)$ be a partition of n. Put $K_m^+ = K_m \cap U_{\underline{n}}$, $K_m^0 = K_m \cap M_{\underline{n}}$, and $K_m^- = K_m \cap \overline{U}_{\underline{n}}$.*

(a) $K_m = K_m^+ K_m^0 K_m^- = K_m^- K_m^0 K_m^+$;

(b) For all $\lambda \in \Lambda^+$ we have $\lambda K_m^+ \lambda^{-1} \subseteq K_m^+$ and $\lambda K_m^- \lambda^{-1} \supseteq K_m^-$;

(c) Let $\underline{n}' = (n_1', \dots, n_s') \leqslant \underline{n}$ be a partition of n and let

$$\lambda \in \Lambda^{++}(M_{\underline{n}'}).$$

Then $\bigcap_i \lambda^i K_m^+ \lambda^{-i} = \{1\} = \bigcap_i \lambda^{-i} K_m^- \lambda^i$ as well as $\bigcup_i \lambda^{-i} K_m^+ \lambda^i = U_{\underline{n}}$ and $\bigcup_i \lambda^i K_m^- \lambda^{-i} = \overline{U}_{\underline{n}}$.

Proof. We first prove (b) and (c) for K_m^+; the result for K_m^- is obtained by passing to transpose matrices. Let $\lambda = \operatorname{diag}(\varpi^{m_1}, \dots, \varpi^{m_n}) \in \Lambda^+$ and $A = E_n + (a_{ij})_{1 \leqslant i < j \leqslant n} \in K_m^+$. We compute

$$\lambda A \lambda^{-1} = E_n + (\varpi^{m_i - m_j} a_{ij})_{i<j} \in E_n + \varpi^m \operatorname{Mat}_{n,n}(o_F).$$

This implies (b).

Let now $\lambda = \operatorname{diag}(\varpi^{m_1} E_{n_1'}, \dots, \varpi^{m_s} E_{n_s'}) \in \Lambda^{++}(M_{\underline{n}'})$. Let $A = E_n + (A_{ij})_{1 \leqslant i < j \leqslant s} \in K_m^+$, where $A_{ij} \in \varpi^m \operatorname{Mat}_{n_i', n_j'}(o_F)$ (this is possible, because $\underline{n}' \leqslant \underline{n}$). For all $l \in \mathbb{Z}$ we compute

$$\lambda^l A \lambda^{-l} = E_n + \big(\varpi^{l(m_i - m_j)} A_{ij}\big)_{i<j} \in E_n + \varpi^{m+l} \operatorname{Mat}_{n,n}(o_F).$$

Then $\bigcap_{l \geqslant 0} \lambda^l K_m^+ \lambda^{-l} \subseteq \bigcap_{l \geqslant 0} K_{m+l} = \{1\}$. It remains to show that each element $A = E_n + (A_{ij})_{1 \leqslant i < j \leqslant s}$ of $U_{\underline{n}}$ lies in some $\lambda^{-l} K_m^+ \lambda^l$. We find $k \in \mathbb{Z}$ such that $A_{ij} \in \varpi^k \operatorname{Mat}_{n_i', n_j'}(o_F)$ for all $i < j$. Then $\lambda^{m-k} A \lambda^{-(m-k)} \in K_m \cap U_{\underline{n}} = K_m^+$, which proves (c).

We now prove (a). Let $A = (A_{ij})_{1 \leqslant i,j \leqslant r} \in K_m$ with $A_{ij} \in \varpi^m \operatorname{Mat}_{n_i, n_j}(o_F)$ for $i \neq j$ and $A_{ii} \in E_{n_i} + \varpi^m \operatorname{Mat}_{n_i, n_i}(o_F)$ for all i. We compute

$$\left(\begin{array}{c|ccc} E_{n_1} & 0 & \cdots & 0 \\ \hline -A_{21}A_{11}^{-1} & E_{n_2} & & \\ \vdots & & \ddots & \\ -A_{r1}A_{11}^{-1} & & & E_{n_r} \end{array}\right) \left(\begin{array}{c|c} A_{11} & A_{12} \cdots A_{1r} \\ \hline A_{21} & \\ \vdots & * \\ A_{r1} & \end{array}\right) \left(\begin{array}{c|ccc} E_{n_1} & -A_{11}^{-1}A_{12} & \cdots & -A_{11}^{-1}A_{1r} \\ \hline 0 & E_{n_2} & & \\ \vdots & & \ddots & \\ 0 & & & E_{n_r} \end{array}\right)$$

$$= \left(\begin{array}{c|c} A_{11} & 0 \cdots 0 \\ \hline 0 & \\ \vdots & A' \\ 0 & \end{array}\right) \in K_m^- A K_m^+.$$

Proceeding by induction, we find $A \in K_m^- K_m^0 K_m^+$. Hence, $K_m = K_m^- K_m^0 K_m^+$. Passing to the inverses also shows $K_m = K_m^+ K_m^0 K_m^-$. □

Remark 12.16. Proposition 12.15(b) shows that $U = U_{(1,\dots,1)}$ is the union of its compact open subgroups. By Exercise 6.5, the modulus character of U (and any of its closed subgroups) is trivial.

Definition 12.17. Let $\underline{n} = (n_1, \dots, n_r)$ be a partition of n. Let $M = M_{\underline{n}} \cong \prod_{s=1}^r \mathrm{GL}_{n_s}(F)$ be the corresponding Levi subgroup of G. Let $\det_{\underline{n}} \colon \prod_s \mathrm{GL}_{n_s}(F) \xrightarrow{\prod_s \det} \prod_s F^\times$. We put

$$M^0 := \det_{\underline{n}}^{-1}\Big(\prod_s o_F^\times\Big) = \prod_{s=1}^r \mathrm{GL}_{n_s}(F)^0, \qquad \text{where } \mathrm{GL}_{n_s}(F)^0 = \det^{-1}(o_F^\times).$$

Let $Z(M) \cong \prod_{s=1}^r F^\times$ be the *center* of M. We make the following easy observations:

- Every compact subgroup H of M is contained in M^0, because $\det_{\underline{n}}(H)$ is a compact subgroup of $\prod_s F^\times \cong \prod_s(\varpi^{\mathbb{Z}} \times o_F^\times)$ and hence contained in $\prod_s o_F^\times$.
- M^0 is a normal subgroup of M, and $M/M^0 \cong \prod_s F^\times/o_F^\times \cong \mathbb{Z}^r$ and $M/Z(M)M^0 \cong \prod_{s=1}^r \mathbb{Z}/n_s\mathbb{Z}$ (for the latter it suffices to observe $\det(Z(\mathrm{GL}_{n_s}(F))\,\mathrm{GL}_{n_s}(F)^0) = (F^\times)^{n_s} o_F^\times = \varpi^{n_s\mathbb{Z}} \times o_F^\times$ for all s).

Proposition 12.18. *Let M be a Levi subgroup of G.*

(a) The subgroup $\mathrm{SL}_n(F) := \det^{-1}(\{1\}) \subseteq \mathrm{GL}_n(F)$ is generated as a group by U and $\overline{U}$.

(b) M^0 is generated by all compact subgroups of M.

(c) M^0 and M are unimodular.

Proof. By Gauß' algorithm, it is clear that $\mathrm{SL}_n(F)$ is generated by U, $\overline{U}$, and $T' := T \cap \mathrm{SL}_n(F)$. Note that for each $t = \mathrm{diag}(t_1, t_2, \dots, t_n) \in T'$ we have

$$t = \prod_{i=1}^{n-1} \mathrm{diag}\big(1, \dots, 1, \overset{\substack{i\\ \downarrow}}{s_i}, s_i^{-1}, 1, \dots, 1\big),$$

where $s_i = t_1 \cdots t_i$ for all $i = 1, \dots, n-1$. We are therefore reduced to the case $n = 2$ and have to show that $\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}$ lies in the group generated by U and $\overline{U}$. For $t = 1$ this is trivial, and for $t \neq 1$ we have

$$\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ t^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1-t^{-1} \\ 0 & 1 \end{pmatrix}.$$

This proves (a). Since every element of U and $\overline{U}$ is contained in a compact subgroup (Remark 12.16), part (b) follows from (a) and the fact that $M^0 = \prod_{i=1}^r \mathrm{GL}_{n_i}(o_F)\,\mathrm{SL}_{n_i}(F)$.

Let $\delta_M \colon M \to \mathbb{R}_{>0}^\times$ be the modulus character of M. Since M^0 is generated by its compact subgroups, we have $\delta_M(M^0) = \{1\}$. It is clear that $\delta_M(Z(M)) = \{1\}$. Hence δ_M factors through a character $\prod_{i=1}^r \mathbb{Z}/n_i\mathbb{Z} \cong M/Z(M)M^0 \to \mathbb{R}_{>0}^\times$, which is trivial since $\mathbb{R}_{>0}^\times$ contains no non-trivial finite subgroups. Hence $\delta_M \equiv 1$. □

Exercise. Let $(V, \pi) \in \mathrm{Rep}(\mathrm{GL}_n(F))$ be a finite dimensional irreducible smooth representation. Show that $\pi = \chi \circ \det$, where $\chi \colon F^\times \to \mathbb{C}^\times$ is a smooth character. (Hint: Use Proposition 12.15 to show that U and $\overline{U}$ act trivially on V.)

§13. The structure of $\mathcal{H}(G, K_m)$

From now on, we assume $G = M_{\underline{n}} \subseteq \mathrm{GL}_n(F)$ for some partition $\underline{n} = (n_1, \ldots, n_r)$ of n. Put $K = \mathrm{GL}_n(o_F) \cap G$.

Our next goal will be to show that every irreducible smooth representation of G is admissible. Since the m-th congruence subgroups $K_m = (E_n + \varpi^m \operatorname{Mat}_{n,n}(o_F)) \cap G$ form a fundamental basis of open compact subgroups, it suffices to show that for every irreducible $(V, \pi) \in \operatorname{Rep}(G)$ the space V^{K_m} is finite dimensional, for all $m \in \mathbb{Z}_{\geqslant 1}$. By Theorem 7.9 we have to show that for all $m \in \mathbb{Z}_{\geqslant 1}$ the simple $\mathcal{H}(G, K_m)$-modules have finite dimension over $\mathbb{C}$. We thus need to study the structure of $\mathcal{H}(G, K_m)$.

We fix a left Haar measure μ_G on G.

Lemma 13.1. *For $g \in G$ we put*

$$c_g := e_{K_m g K_m} = \operatorname{vol}(K_m g K_m; \mu_G)^{-1} \cdot \mathbf{1}_{K_m g K_m}.$$

The set $\{c_g\}_{g \in K_m \backslash G / K_m}$ is a $\mathbb{C}$-basis of $\mathcal{H}(G, K_m)$. Moreover, if $K_m g K_m g' K_m = K_m g g' K_m$, then $c_g * c_{g'} = c_{gg'}$.

Proof. The first assertion is clear from Proposition 7.4. Assume now that $K_m g K_m g' K_m = K_m g g' K_m$. Let $h \in G$. The map $x \mapsto \mathbf{1}_{K_m g K_m}(x) \cdot \mathbf{1}_{K_m g' K_m}(x^{-1}h)$ is the characteristic function $\mathbf{1}_{K_m g K_m \cap h K_m g'^{-1} K_m}$; it is non-zero precisely when $h \in K_m g K_m g' K_m$. Hence, we have

$$(\mathbf{1}_{K_m g K_m} * \mathbf{1}_{K_m g' K_m})(h) = \operatorname{vol}\big(K_m g K_m \cap h K_m g'^{-1} K_m\big).$$

Write $K_m g K_m = \bigsqcup_{i=1}^{d_g} g_i K_m$ and $K_m g' K_m = \bigsqcup_{j=1}^{d_{g'}} g'_j K_m$, where $d_g \operatorname{vol}(K_m) = \operatorname{vol}(K_m g K_m)$ and $d_{g'} \operatorname{vol}(K_m) = \operatorname{vol}(K_m g' K_m)$ because μ_G is left invariant. Observe $h K_m g'^{-1} K_m = \bigsqcup_{j=1}^{d_{g'}} h K_m g_j'^{-1}$. By counting the left cosets in $K_m g K_m \cap h K_m g'^{-1} K_m$, we compute

$$\begin{aligned}
\operatorname{vol}\big(K_m g K_m \cap h K_m g'^{-1} K_m\big) &\cdot \operatorname{vol}(K_m h K_m) \\
&= \#\left\{ i \,\middle|\, g_i \in h K_m g_j'^{-1} \text{ for some } j \right\} \cdot \operatorname{vol}(K_m h K_m) \operatorname{vol}(K_m) \\
&= \#\left\{ (i,j) \,\middle|\, g_i g'_j \in h K_m \right\} \cdot \operatorname{vol}(K_m h K_m) \operatorname{vol}(K_m) \\
&= \#\left\{ (i,j) \,\middle|\, g_i g'_j \in K_m h K_m \right\} \cdot \operatorname{vol}(K_m)^2 \\
&= d_g d_{g'} \cdot \operatorname{vol}(K_m)^2 \\
&= \operatorname{vol}(K_m g K_m) \cdot \operatorname{vol}(K_m g' K_m).
\end{aligned}$$

Here, the third equality uses the fact that $\operatorname{vol}\big(K_m g K_m \cap h K_m g'^{-1} K_m\big)$ only depends on the double coset $K_m h K_m$, and the fourth equality uses $K_m g K_m g' K_m = K_m g g' K_m$. Finally, we have

$$\begin{aligned}
(c_g * c_{g'})(h) &= \frac{\operatorname{vol}(K_m g K_m \cap h K_m g'^{-1} K_m)}{\operatorname{vol}(K_m g K_m) \operatorname{vol}(K_m g' K_m)} \\
&= \frac{1}{\operatorname{vol}(K_m h K_m)} \cdot \mathbf{1}_{K_m g g' K_m}(h) = c_{gg'}(h)
\end{aligned}$$

for all $h \in G$. ∎

Recall that K_m is a normal subgroup of K. Hence $\mathcal{H}(K, K_m)$ is a subalgebra of $\mathcal{H}(G, K_m)$ of dimension $[K : K_m]$.

Theorem 13.2. *Put $C := \langle c_\lambda \mid \lambda \in \Lambda^+(G)\rangle \subseteq \mathcal{H}(G, K_m)$, where $\Lambda^+(G) := \prod_{i=1}^r \Lambda^+(\mathrm{GL}_{n_i}(F))$. Then:*

(a) $\mathcal{H}(G, K_m) = \mathcal{H}(K, K_m) C\, \mathcal{H}(K, K_m)$.

(b) C is a commutative, finitely generated algebra. In fact, we have

$$c_{\lambda\lambda'} = c_\lambda * c_{\lambda'} \qquad \text{for all } \lambda, \lambda' \in \Lambda^+. \tag{3.1}$$

Proof. Let $g \in G$. By the Cartan decomposition 12.4 applied to each factor of G, we find $k, k' \in K$ and $\lambda \in \Lambda^+(G)$ with $g = k\lambda k'$. We have

$$K_m k\lambda K_m = K_m k K_m \lambda K_m \qquad \text{and} \qquad K_m k\lambda k' K_m = K_m k\lambda K_m k' K_m,$$

because $K_m \subseteq K$ is normal. By Lemma 13.1, we have

$$c_k * c_\lambda * c_{k'} = c_{k\lambda} * c_{k'} = c_{k\lambda k'} = c_g.$$

This proves (a). We now prove (b). Note that each $\Lambda^+(\mathrm{GL}_{n_s}(F))$ is generated as a commutative monoid by the elements

$$\lambda_{s,i} := \mathrm{diag}(\underbrace{\varpi, \ldots, \varpi}_{i \text{ times}}, 1, \ldots, 1) \in \mathrm{GL}_{n_s}(F) \subseteq \mathrm{GL}_n(F), \tag{3.2}$$

for $1 \leqslant i \leqslant n_s$, and λ_{s,n_s}^{-1}. To finish the proof, it remains to show (3.1). Again by Lemma 13.1 it suffices to show

$$K_m \lambda K_m \lambda' K_m = K_m \lambda\lambda' K_m. \tag{3.3}$$

Applying Proposition 12.15 for $\underline{n} = (1, \ldots, 1)$ to each factor of G, we have

$$\lambda K_m \lambda' = \lambda K_m^+ K_m^0 K_m^- \lambda' = (\lambda K_m^+ \lambda^{-1}) \cdot \lambda K_m^0 \lambda' \cdot (\lambda'^{-1} K_m^- \lambda') \subseteq K_m \lambda\lambda' K_m.$$

(Note that $K_m^0 \subseteq T$ and $\lambda, \lambda' \in T$, and T is commutative.) We deduce "$\subseteq$" in (3.3). The other inclusion is trivial. $\square$

Fix $\lambda \in \Lambda^+(G)$. For each $(V, \pi) \in \mathrm{Rep}(G)$ we are going to describe the kernel of the maps

$$\pi(c_{\lambda^l}) \colon V^{K_m} \longrightarrow V^{K_m}, \qquad \text{for } l \in \mathbb{Z}_{\geqslant 0}.$$

Let $\underline{n}' \leqslant \underline{n}$ be the unique partition for which $\lambda \in \Lambda^{++}(M_{\underline{n}'}, G)$ (see Notation 12.14). Put $N := U_{\underline{n}'} \cap G$. Recall the *Jacquet functor* from Exercise 9.4(b): It is the functor

$$\begin{aligned} J_N \colon \mathrm{Rep}(P_{\underline{n}'} \cap G) &\longrightarrow \mathrm{Rep}(M_{\underline{n}'}), \\ (W, \sigma) &\longmapsto (W_N, J_N(\sigma)), \end{aligned}$$

where we set $W_N := W/W(N)$ and $W(N) = \langle w - \sigma(x)w \mid x \in N, w \in W\rangle$.

Proposition 13.3. *Let* $(V, \pi) \in \mathrm{Rep}(G)$. *Then*

$$\bigcup_{l \geqslant 0} \operatorname{Ker} \pi(c_{\lambda^l}) \cap V^{K_m} = V(N) \cap V^{K_m}.$$

Proof. By Proposition 12.15(c) (applied with $\underline{n}'$), we have $N = \bigcup_{l \geqslant 0} N_l$, where each $N_l := \lambda^{-l} K_m^+ \lambda^l$ is a compact open subgroup of N. Note that $V(N) = \bigcup_{l \geqslant 0} V(N_l)$. By Lemma 7.8 we have $V(N_l) = \operatorname{Ker} \pi_{|N}(e_{N_l})$. Hence, given any $v \in V^{K_m}$, we have to show

$$\pi(c_{\lambda^l})v = 0 \iff \pi_{|N}(e_{N_l})v = 0. \tag{3.4}$$

Write $\lambda^{-l} K_m^+ \lambda^l = \bigsqcup_{i=1}^d u_i K_m^+$. By Proposition 12.15 we have $K_m = K_m^+ K_m^0 K_m^-$ and $\lambda^{-l} K_m^0 K_m^- \lambda^l \subseteq K_m$. Now, observe that

$$K_m \lambda^l K_m = \lambda^l \cdot \lambda^{-l} K_m \lambda^l K_m = \lambda^l \cdot \lambda^{-l} K_m^+ \lambda^l K_m = \bigsqcup_{i=1}^d \lambda^l u_i K_m$$

is a disjoint union (if $u_i \in u_j K_m$, then $u_j^{-1} u_i \in K_m \cap N = K_m^+$, hence $u_i = u_j$). We now compute

$$\begin{aligned}\pi(c_{\lambda^l})v &= \frac{1}{d} \sum_{i=1}^d \pi(\lambda^l u_i)v = \frac{1}{d} \cdot \pi(\lambda^l) \sum_{i=1}^d \pi(u_i)v \\ &= \pi(\lambda^l) \pi_{|N}(e_{N_l})v.\end{aligned}$$

As $\pi(\lambda^l)$ is an isomorphism, this shows (3.4), which finishes the proof. □

The last result suggests that we should look at the Jacquet functor J_N in more detail.

§14. Parabolic Induction and Parabolic Restriction

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \ldots, n_r)$ of n. We fix a standard parabolic subgroup $P = MN$ with Levi subgroup M and unipotent radical N, corresponding to some partition $\underline{n}' \leqslant \underline{n}$. (This means $P = P_{\underline{n}'} \cap G$, $M = M_{\underline{n}'} \cap G$ and $N = U_{\underline{n}'} \cap G$.) Recall the modulus character $\delta_P \colon P \to \mathbb{R}_{>0}^\times$, which is given as follows: Choose any compact open subgroup $H \subseteq P$; then $\delta_P(g) = [gHg^{-1} : H]$ (generalized index) for each $g \in P$. See Exercise 6.5. As $\mathbb{R}_{>0}$ admits unique square roots, the character

$$\begin{aligned}\delta_P^{1/2} \colon P &\longrightarrow \mathbb{R}_{>0}^\times, \\ g &\longmapsto \sqrt{\delta_P(g)}\end{aligned}$$

is well-defined. We denote $\delta_P^{-1/2}$ the inverse of $\delta_P^{1/2}$.

Lemma 14.1. *One has* $(\delta_P)_{|N} \equiv 1$. *For all* $m \in M$ *one has*

$$\delta_P(m) = [m K_N m^{-1} : K_N], \qquad \textit{where } K_N = K_1 \cap N.$$

Proof. By Proposition 12.15, every $u \in N$ is contained in a compact subgroup of P. Hence, by Exercise 6.5 we have $\delta_P(u) = 1$.

In order to prove the last assertion, we use the following general

Fact. Let P be a topological group, and let $N, M \subseteq P$ be closed subgroups such that $N \trianglelefteq P$ is normal and the composite $M \hookrightarrow P \twoheadrightarrow P/N$ is an isomorphism. Let $H' \subseteq H \subseteq P$ be compact open subgroups. Put $H_M \coloneqq H \cap M$ and $H_N \coloneqq H \cap N$, and similarly for H'_M and H'_N. Suppose that $H = H_M H_N$ and $H' = H'_M H'_N$. Then

$$[H : H'] = [H_M : H'_M] \cdot [H_N : H'_N].$$

Proof of the Fact. Write $H_M = \bigsqcup_{i=1}^{d_M} m_i H'_M$ and $H_N = \bigsqcup_{j=1}^{d_N} u_j H'_N$, so that $d_M = [H_M : H'_M]$ and $d_N = [H_N : H'_N]$. The claim amounts to showing that we have a disjoint union

$$H = \bigsqcup_{i=1}^{d_M} \bigsqcup_{j=1}^{d_N} m_i u_j H'. \tag{3.5}$$

Note that H'_M normalizes H_N and H'_N. Hence

$$H = H_M H_N = \bigcup_i m_i H'_M H_N = \bigcup_i m_i H_N H'_M = \bigcup_{i,j} m_i u_j H'_N H'_M = \bigcup_{i,j} m_i u_j H'.$$

It remains to prove that (3.5) is disjoint. Suppose $m_i u_j H' = m_{i'} u_{j'} H'$. Fix $h' \in H'$ with $m_i u_j = m_{i'} u_{j'} h'$, and write $h' = h'_M h'_N$ with $h'_M \in H'_M$ and $h'_N \in H'_N$. Then

$$m_i \cdot u_j = m_{i'} u_{j'} h'_M h'_N = (m_{i'} h'_M) \cdot \big(h'_M u_{j'} h'^{-1}_M h'_N\big), \qquad \text{in } H_M H_N.$$

Since $M \cap N = \{1\}$, we have $m_i = m_{i'} h'_M$ and $u_j = h'_M u_{j'} h'^{-1}_M h'_N$. But by assumption, we have $i = i'$ and $h'_M = 1$, and from the resulting equality $u_j = u_{j'} h'_N$ we deduce $j = j'$. Hence, the union in (3.5) is indeed disjoint. □

Write $K_P = K_1 \cap P$ and $K_M = K_1 \cap M$. Then $K_P = K_M K_N$, and for each $m \in M$ we have $m K_P m^{-1} = (m K_M m^{-1}) \cdot (m K_N m^{-1})$ and $K'_P \coloneqq K_P \cap m K_P m^{-1} = K'_M \cdot K'_N$, where $K'_M \coloneqq K_M \cap m K_M m^{-1}$ and $K'_N \coloneqq K_N \cap m K_N m^{-1}$. Note that $M \cong \mathrm{GL}_{n'_1}(F) \times \cdots \times \mathrm{GL}_{n'_{r'}}(F)$ is unimodular by Proposition 12.18. We now compute

$$\begin{aligned}\delta_P(m) = \frac{[m K_P m^{-1} : K'_P]}{[K_P : K'_P]} &= \frac{[m K_M m^{-1} : K'_M] \cdot [m K_N m^{-1} : K'_N]}{[K_M : K'_M] \cdot [K_N : K'_N]} \\ &= \delta_M(m) \cdot [m K_N m^{-1} : K_N] = [m K_N m^{-1} : K_N].\end{aligned}$$

□

Definition 14.2. (a) For $(W, \sigma) \in \mathrm{Rep}(M)$, the representation

$$\begin{aligned}\boldsymbol{i}_P^G(W, \sigma) &\coloneqq \mathrm{Ind}_P^G\big(W, \delta_P^{1/2} \otimes \mathrm{Inf}_P^M \sigma\big) \\ &= \left\{ f\colon G \to W \;\middle|\; \begin{array}{l} \exists H \subseteq G \text{ compact open such that} \\ f(gh) = f(g) \text{ for all } g \in G,\, h \in H, \text{ and} \\ f(xg) = \delta_P^{1/2}(x)\sigma(x) f(g) \text{ for all } g \in G,\, x \in P \end{array} \right\}\end{aligned}$$

is smooth, where G acts on $f \in \boldsymbol{i}_P^G(W, \sigma)$ by right translation: $(gf)(g') = f(g'g)$ for all $g, g' \in G$. We call $\boldsymbol{i}_P^G(W, \sigma)$ the representation *parabolically induced* from (W, σ).

(b) For $(V,\pi) \in \mathrm{Rep}(G)$, the M-representation

$$\boldsymbol{r}_P^G(V,\pi) := \bigl(V_N, J_N(\delta_P^{-1/2} \otimes \pi_{|P})\bigr)$$

is smooth. (See also Exercise 9.4). We call $\boldsymbol{r}_P^G(V,\pi)$ the representation *parabolically restricted* from (V,π).

The parabolic induction functor $\boldsymbol{i}_P^G\colon \mathrm{Rep}(M) \to \mathrm{Rep}(G)$ allows us to construct new smooth G-representations from smooth representations of the "smaller" group M. This breaks up the classification of irreducible smooth representations into two steps:

(i) Classify the irreducible smooth representations arising as a subquotient of $\boldsymbol{i}_P^G(W,\sigma)$ for some parabolic subgroup $P = MN$ of G and some $(W,\sigma) \in \mathrm{Rep}(M)$.

(ii) Classify the irreducible smooth representations which are not a subquotient of a parabolically induced representation.

The representations falling into case (ii), called *supercuspidal*, are to be thought of as the "building blocks" of smooth representations in the sense that knowledge of the supercuspidal representations (of all Levi subgroups of G) and of the parabolic induction functors provides a complete understanding of *all* irreducible smooth representations.

The following properties of $\boldsymbol{i}_P^G$ and $\boldsymbol{r}_P^G$ will be essential in the following:

Theorem 14.3. *Let $P = MN$ be a standard parabolic subgroup of G corresponding to a partition $\underline{n}' \leqslant \underline{n}$. Let $(V,\pi) \in \mathrm{Rep}(G)$ and $(W,\sigma) \in \mathrm{Rep}(M)$.*

(a) There is a natural $\mathbb{C}$-linear isomorphism

$$\mathrm{Hom}_M\bigl(\boldsymbol{r}_P^G(V,\pi),(W,\sigma)\bigr) \cong \mathrm{Hom}_G\bigl((V,\pi),\boldsymbol{i}_P^G(W,\sigma)\bigr).$$

In other words: $\boldsymbol{r}_P^G$ is left adjoint to $\boldsymbol{i}_P^G$.

(b) The functors $\boldsymbol{i}_P^G$ and $\boldsymbol{r}_P^G$ are exact.

(c) If $(V,\pi) \in \mathrm{Rep}(G)$ is finitely generated, then $\boldsymbol{r}_P^G(V,\pi) \in \mathrm{Rep}(M)$ is finitely generated.

(d) If (W,σ) is admissible, then $\boldsymbol{i}_P^G(W,\sigma)$ is admissible.

(e) $\boldsymbol{i}_P^G$ and $\boldsymbol{r}_P^G$ are transitive. *More concretely, let $\underline{n}' \leqslant \underline{n}'' \leqslant \underline{n}$ be another partition. Put $Q = P_{\underline{n}''} \cap G$ and $L = M_{\underline{n}''}$ so that $Q \supseteq P$ and $L \supseteq M$. Then there are natural isomorphisms*

$$\boldsymbol{i}_P^G(W,\sigma) \cong \boldsymbol{i}_Q^G\,\boldsymbol{i}_{P\cap L}^L(W,\sigma) \qquad \text{and} \qquad \boldsymbol{r}_P^G(V,\pi) \cong \boldsymbol{r}_{P\cap L}^L\,\boldsymbol{r}_Q^G(V,\pi).$$

Proof. For (a) we compute

$$\begin{aligned}
\mathrm{Hom}_M\bigl(\boldsymbol{r}_P^G\,\pi,\sigma\bigr) &= \mathrm{Hom}_M\bigl(J_N(\delta_P^{-1/2} \otimes \pi_{|P}),\sigma\bigr) \\
&\cong \mathrm{Hom}_P\bigl(\delta_P^{-1/2} \otimes \pi_{|P}, \mathrm{Inf}_P^M\,\sigma\bigr) && \text{(Exercise 9.4(b))} \\
&= \mathrm{Hom}_P\bigl(\pi_{|P}, \delta_P^{1/2} \otimes \mathrm{Inf}_P^M\,\sigma\bigr) \\
&\cong \mathrm{Hom}_G\bigl(\pi, \mathrm{Ind}_P^G(\delta_P^{1/2} \otimes \mathrm{Inf}_P^M\,\sigma)\bigr) && \text{(Proposition 9.3)} \\
&= \mathrm{Hom}_G\bigl(\pi, \boldsymbol{i}_P^G\,\sigma\bigr).
\end{aligned}$$

We now prove (b). For the exactness of $\boldsymbol{r}_P^G = J_N \circ (\delta_P^{-1/2} \otimes \_) \circ \operatorname{Res}_P^G$, we note that the functors $\delta_P^{-1/2} \otimes \_$ and Res_P^G are exact. It remains to show that $J_N\colon \operatorname{Rep}(P) \to \operatorname{Rep}(M)$ is exact. Let

$$(V', \pi') \xrightarrow{\varphi} (V, \pi) \xrightarrow{\psi} (V'', \pi'') \tag{3.6}$$

be an exact sequence in $\operatorname{Rep}(P)$. We have to show that $J_N(V') \xrightarrow{J_N(\varphi)} J_N(V) \xrightarrow{J_N(\psi)} J_N(V'')$ is exact, that is, $\operatorname{Im}(J_N(\varphi)) = \operatorname{Ker}(J_N(\psi))$. We have $J_N(\psi) \circ J_N(\varphi) = J_N(\psi \circ \varphi) = 0$, which shows "$\subseteq$". For the reverse inclusion, let $v \in V$ such that $J_N(\psi)(v + V(N)) = 0$. This means $\psi(v) \in V''(N)$. By Proposition 12.15, N is an increasing union of compact open subgroups. Hence, there exists a compact open subgroup $H \subseteq N$ such that $\psi(v) \in V''(H)$. In other words: $J_H(\psi)(v + V(H)) = 0$. By Lemma 7.8 we have $J_H = \pi_{|N}(e_H) = (\_)^H$, which is exact by Lemma 5.8. Hence, there exists $v' \in V'$ such that $J_H(\varphi)(v' + V'(H)) = v + V(H)$, that is, $\varphi(v') - v \in V(H)$. As $V(H) \subseteq V(N)$, this implies $\varphi(v') - v \in V(N)$ and hence $J_N(\varphi)(v' + V'(N)) = v + V(N)$. This shows $\operatorname{Im}(J_N(\varphi)) = \operatorname{Ker}(J_N(\psi))$.

We now prove that $\boldsymbol{i}_P^G = \operatorname{Ind}_P^G \circ (\delta_P^{1/2} \otimes \_) \circ \operatorname{Inf}_P^M$ is exact. We observe that $\delta_P^{1/2} \otimes \_$ and Inf_P^M are exact. It remains to show that $\operatorname{Ind}_P^G\colon \operatorname{Rep}(P) \to \operatorname{Rep}(G)$ is exact. Consider an exact sequence as in (3.6). We have to show that $\operatorname{Ind}_P^G V' \xrightarrow{\operatorname{Ind}_P^G \varphi} \operatorname{Ind}_P^G V \xrightarrow{\operatorname{Ind}_P^G \psi} \operatorname{Ind}_P^G V''$ is exact. By Exercise 5.9 it suffices to show that, given any compact open subgroup $H \subseteq G$, the induced sequence

$$(\operatorname{Ind}_P^G V')^H \longrightarrow (\operatorname{Ind}_P^G V)^H \longrightarrow (\operatorname{Ind}_P^G V'')^H \tag{3.7}$$

is exact. By the Mackey decomposition (Proposition 9.5) we have

$$\begin{aligned}(\operatorname{Ind}_P^G V)^H &\cong \Big(\prod_{g \in P\backslash G/H} \operatorname{Ind}_{g^{-1}Pg \cap H}^H g_*^{-1} V\Big)^H = \prod_{g \in P\backslash G/H} (\operatorname{Ind}_{g^{-1}Pg \cap H}^H g_*^{-1} V)^H \\ &\cong \prod_{g \in P\backslash G/H} (g_*^{-1} V)^{g^{-1}Pg \cap H} = \prod_{g \in P\backslash G/H} V^{P \cap gHg^{-1}},\end{aligned} \tag{3.8}$$

where the second isomorphism is an instance of Frobenius reciprocity (Proposition 9.3); similarly for $(\operatorname{Ind}_P^G V')^H$ and $(\operatorname{Ind}_P^G V'')^H$. Now, the sequence (3.7) becomes

$$\prod_{g \in P\backslash G/H} (V')^{P \cap gHg^{-1}} \longrightarrow \prod_{g \in P\backslash G/H} V^{P \cap gHg^{-1}} \longrightarrow \prod_{g \in P\backslash G/H} (V'')^{P \cap gHg^{-1}}$$

which is exact, because $(\_)^{P \cap gHg^{-1}}$ is exact by Lemma 5.8.[1] This shows that Ind_P^G is exact.

We prove (c). Let $v_1, \dots, v_d \in V$ which generate (V, π) as a G-representation. Fix a compact open subgroup $H \subseteq G$ such that $v_1, \dots, v_d \in V^H$. By the Iwasawa decomposition 12.7, the space $P\backslash G$ is compact and hence $P\backslash G/H$ is finite. Let $g_1, \dots, g_k$ be a representing system for $P\backslash G/H$. Then $\{\pi(g_i)v_j \mid 1 \leqslant i \leqslant k, 1 \leqslant j \leqslant d\}$ generates $(V, \delta_P^{-1/2} \otimes \pi_{|P})$ as a P-representation. But then $\{\pi(g_i)v_j + V(N)\}_{i,j}$ generate $\boldsymbol{r}_P^G(V, \pi) = (V_N, J_N(\delta_P^{-1/2} \otimes \pi_{|P}))$ as an M-representation.

We now prove (d). Assume $(W, \sigma) \in \operatorname{Rep}(M)$ is admissible. Let $H \subseteq G$ be a compact open subgroup. We have to show that $(\operatorname{Ind}_P^G W)^H$ is finite dimensional. Since clearly $(W, \delta_P^{1/2} \otimes \operatorname{Inf}_P^M \sigma)$

[1] Note that $P\backslash G/H$ is finite by the Iwasawa decomposition 12.7, and hence the products are finite. But this is not needed here.

is admissible, the spaces $W^{P\cap gHg^{-1}}$ are finite dimensional for all $g \in G$. By the Iwasawa decomposition 12.7, $P\backslash G/H$ is finite. Now, (3.8) shows that $(\operatorname{Ind}_P^G W)^H$ is a finite product of finite dimensional vector spaces, hence itself finite dimensional. Thus, $\boldsymbol{i}_P^G(W,\sigma)$ is admissible.

Finally, we prove (e). Let $Q \supseteq P$ be a parabolic subgroup of G with Levi L and unipotent radical R. Note that $N \subseteq Q = LR$ and $R \trianglelefteq N$ is a normal subgroup, so that we have $N = (N \cap L)\cdot R$. For every $m \in M$ we have by Lemma 14.1 (and the fact in its proof)

$$\delta_P(m) = [mK_Nm^{-1} : K_N] = [mK_{N\cap L}m^{-1} : K_{N\cap L}]\cdot[mK_Rm^{-1} : K_R] = \delta_{P\cap L}(m)\cdot\delta_Q(m).$$

Let $(V,\pi) \in \operatorname{Rep}(G)$. We have to show that the maps

$$\begin{array}{ccc} & V & \\ {}^{f_1}\swarrow & & \searrow^{f_2} \\ J_N(V) & \longleftarrow\dashrightarrow & J_{N\cap L}(J_R(V)) \end{array}$$

factor through the dashed isomorphism. The kernel of f_1 is $V(N)$ and the kernel of f_2 is given by $V(N\cap L) + V(R)$. Since $N = (N\cap L)\cdot R$, we have $V(N\cap L) + V(R) = V(N)$. [Indeed, "$\subseteq$" is clear, and for each $u = xy \in N$ with $x \in N \cap L$ and $y \in R$, we have $v - \pi(u)v = (v - \pi(y)v) + (\pi(y)v - \pi(x)\pi(y)v) \in V(N\cap L) + V(R)$, which shows "$\supseteq$".] We thus have a canonical isomorphism

$$J_N(V) \xrightarrow{\cong} J_{N\cap L}\big(J_R(V)\big)$$

given by $v + V(N) \mapsto (v + V(R)) + J_R(V)(N\cap L)$. Since also $\delta_P(m) = \delta_{P\cap L}(m)\cdot\delta_Q(m)$ for all $m \in M$, this isomorphism induces the canonical isomorphism $\boldsymbol{r}_P^G\,\pi \xrightarrow{\cong} \boldsymbol{r}_{P\cap L}^L\,\boldsymbol{r}_Q^G\,\pi$.

Now, for all $(W,\sigma) \in \operatorname{Rep}(M)$ and $(V,\pi) \in \operatorname{Rep}(G)$ we have by (a) natural isomorphisms

$$\begin{aligned} \operatorname{Hom}_G\big(\pi, \boldsymbol{i}_Q^G\,\boldsymbol{i}_{P\cap L}^L\,\sigma\big) &\cong \operatorname{Hom}_L\big(\boldsymbol{r}_Q^G\,\pi, \boldsymbol{i}_{P\cap L}^L\,\sigma\big) \cong \operatorname{Hom}_M\big(\boldsymbol{r}_{P\cap L}^L\,\boldsymbol{r}_Q^G\,\pi, \sigma\big) \\ &\cong \operatorname{Hom}_M\big(\boldsymbol{r}_P^G\,\pi, \sigma\big) \cong \operatorname{Hom}_G\big(\pi, \boldsymbol{i}_P^G\,\sigma\big) \end{aligned}$$

By the Yoneda lemma below, we deduce a natural isomorphism $\boldsymbol{i}_Q^G\,\boldsymbol{i}_{P\cap L}^L\,\sigma \cong \boldsymbol{i}_P^G\,\sigma$. ☐

Yoneda Lemma 14.4. *Let $\mathscr{A}$ be a category, and fix two objects $A, B \in \mathscr{A}$. Suppose that there is a natural bijection*

$$\alpha_C\colon \operatorname{Hom}_{\mathscr{A}}(C,A) \xrightarrow{\cong} \operatorname{Hom}_{\mathscr{A}}(C,B)$$

for each $C \in \mathscr{A}$. Then $\alpha_A(\mathrm{id}_A)\colon A \to B$ is an isomorphism in $\mathscr{A}$.

Proof. Since α_B is surjective, there exists a morphism $\psi\colon B \to A$ with $\alpha_B(\psi) = \mathrm{id}_B$. The naturality means that for every morphism $\phi\colon C \to C'$ in $\mathscr{A}$ the diagram

$$\begin{array}{ccc} \operatorname{Hom}_{\mathscr{A}}(C',A) & \xrightarrow{\alpha_{C'}} & \operatorname{Hom}_{\mathscr{A}}(C',B) \\ {\scriptstyle\phi^*}\big\downarrow & & \big\downarrow{\scriptstyle\phi^*} \\ \operatorname{Hom}_{\mathscr{A}}(C,A) & \xrightarrow[\alpha_C]{} & \operatorname{Hom}_{\mathscr{A}}(C,B) \end{array}$$

is commutative, *i.e.*, $\alpha_C(f\circ\phi) = \alpha_{C'}(f)\circ\phi$ for all $f\colon C' \to A$.

By naturality we have $\alpha_A(\mathrm{id}_A)\circ\psi = \alpha_B(\mathrm{id}_A\circ\psi) = \alpha_B(\psi) = \mathrm{id}_B$. Conversely, we compute $\alpha_A(\psi\circ\alpha_A(\mathrm{id}_A)) = \alpha_B(\psi)\circ\alpha_A(\mathrm{id}_A) = \mathrm{id}_B\circ\alpha_A(\mathrm{id}_A) = \alpha_A(\mathrm{id}_A)$. As α_A is injective, we deduce $\psi\circ\alpha_A(\mathrm{id}_A) = \mathrm{id}_A$. Hence, $\alpha_A(\mathrm{id}_A)$ is an isomorphism with inverse ψ. ☐

§15. Cuspidal Representations and Uniform Admissibility

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \dots, n_r)$ of n and $K = \mathrm{GL}_n(o_F) \cap G$.

Definition 15.1. A representation $(V, \pi) \in \mathrm{Rep}(G)$ is called *cuspidal* if $\boldsymbol{r}_P^G(V, \pi) = \{0\}$ for every standard parabolic subgroup $P = MN \subsetneq G$.

Note that the condition $\boldsymbol{r}_P^G \pi = \{0\}$ is equivalent to $J_N(V) = \{0\}$ (and hence to $V = V(N)$).

Remark. (a) As $\boldsymbol{r}_P^G$ is exact by Theorem 14.3(b), it follows that every subquotient of a cuspidal representation is cuspidal.

(b) If (V, π) is cuspidal, then $\boldsymbol{r}_P^G(V, \pi) = \{0\}$ for every (not necessarily standard) parabolic subgroup $P = MN \subsetneq G$. Indeed, if $g \in G$ is such that gPg^{-1} is a standard parabolic, then $V = \pi(g^{-1})(V) = \pi(g^{-1})(V(gNg^{-1})) \subseteq V(N)$.

(c) If $(V, \pi) \in \mathrm{Rep}(G)$ satisfies $\boldsymbol{r}_P^G(V, \pi) = \{0\}$ for every maximal parabolic subgroup $P \subsetneq G$, then (V, π) is cuspidal. This follows at once from the fact that $\boldsymbol{r}_P^G$ is transitive (Theorem 14.3(e)).

The following important result makes precise the assertion that the cuspidal representations are the "building blocks" of smooth representations.

Lemma 15.2. *Let $(V, \pi) \in \mathrm{Rep}(G)$ be irreducible. There exists a standard parabolic subgroup $P = MN$ of G and an irreducible cuspidal representation $(W, \tau) \in \mathrm{Rep}(M)$ together with a G-equivariant embedding*

$$(V, \pi) \hookrightarrow \boldsymbol{i}_P^G(W, \tau)$$

Proof. Let $P = MN$ be a minimal standard parabolic subgroup with $J_N(V) \neq \{0\}$. Then $\boldsymbol{r}_P^G \pi$ is cuspidal by Theorem 14.3(e) and the minimality of P. As π is finitely generated, so is $\boldsymbol{r}_P^G \pi$ (Theorem 14.3(c)). Hence, there exists an irreducible quotient $\boldsymbol{r}_P^G \pi \twoheadrightarrow \tau$ in $\mathrm{Rep}(M)$. By Theorem 14.3(a), we obtain a non-zero G-equivariant map $\pi \to \boldsymbol{i}_P^G \tau$, which is injective since π is irreducible. □

Recall the subgroup $G^0 := \det_{\underline{n}}^{-1}(\prod_{i=1}^r o_F^\times) \subseteq G$ and the center $Z = Z(G) \cong \prod_{i=1}^r F^\times$ of G.

Theorem 15.3. *Let $(V, \pi) \in \mathrm{Rep}(G)$. The following are equivalent:*

(a) (V, π) is cuspidal.

(b) The functions $f_{H,v} \colon G \to V$ (Definition 11.2) have compact support modulo $Z(G)$ for all compact open subgroups $H \subseteq G$ and all $v \in V \smallsetminus \{0\}$.

(c) The matrix coefficients of (V, π) (Definition 11.3) have compact support modulo $Z(G)$.

(d) $(V, \pi_{|G^0})$ is compact (Definition 11.2).

Proof. "(a) $\Longrightarrow$ (b)": Let $H \subseteq G$ and $v \in V$ as in (b). Choose $m \geqslant 1$ such that $K_m \subseteq H$ and $v \in V^{K_m}$. Then $\operatorname{Supp} f_{H,v} \subseteq \operatorname{Supp} f_{K_m,v}$ and hence we may assume from the start that $H = K_m$ and $v \in V^H$. Then $f_{H,v}(g) = \pi(e_H)\pi(g^{-1})v = \pi(e_{Hg^{-1}H})v$ for all $g \in G$. Consider the function

$$\begin{aligned} \phi_v \colon \Lambda^+(G) &\longrightarrow V^H, \\ \lambda &\longmapsto \pi(c_\lambda)v = f_{H,v}(\lambda^{-1}), \end{aligned}$$

where $c_\lambda = e_{H\lambda H}$. We have $G = K\Lambda^+(G)K$ by the Cartan decomposition 12.4. Note that $\pi(e_H)\pi(k) = \pi(k)\pi(e_H)$ for all $k \in K$ (as H is normal in K). For any $g = k'\lambda k$ with $k, k' \in K$ and $\lambda \in \Lambda^+(G)$, we have $f_{H,v}(g^{-1}) = \pi(k')f_{H,\pi(k)v}(\lambda^{-1}) = \pi(k')\phi_{\pi(k)v}(\lambda)$ and hence

$$\bigl(\operatorname{Supp} f_{H,v}\bigr)^{-1} \subseteq \bigcup_{k\in K/H} K \operatorname{Supp} \phi_{\pi(k)v} K.$$

It therefore suffices to show that $\operatorname{Supp}\phi_v$ is finite modulo Z, for all $v \in V^H$. Fix any $v \in V^H$. Given $\nu \in \Lambda^+(G) \smallsetminus Z$, let $P_\nu = M_\nu U_\nu$ be the unique (proper) parabolic subgroup of G for which $\nu \in \Lambda^{++}(M_\nu, G)$ (see Notation 12.14). By Proposition 13.3 and since (V, π) is cuspidal, we have

$$V^H = V^H \cap V(U_\nu) = V^H \cap \bigcup_{k\geqslant 0} \operatorname{Ker} \pi(c_{\nu^k}).$$

Hence, there exists $k_\nu \in \mathbb{Z}_{\geqslant 0}$ such that $\phi_v(\nu^k) = 0$ for all $k \geqslant k_\nu$. Recall the elements $\lambda_{s,i}$ from (3.2); for every $\lambda \in \Lambda^+(G)$ we then have $\lambda = \prod_{s=1}^r \prod_{i_s=1}^{n_s} \lambda_{s,i_s}^{d_{s,i_s}(\lambda)}$ for uniquely determined $d_{s,i_s}(\lambda) \in \mathbb{Z}_{\geqslant 0}$, for $1 \leqslant i_s < n_s$, and $d_{s,n_s}(\lambda) \in \mathbb{Z}$. Define $k_0 := \max\bigl\{k_{\lambda_{s,i_s}} \bigm| 1 \leqslant s \leqslant r, 1 \leqslant i_s < n_s\bigr\}$ and then

$$X := \bigl\{\lambda \in \Lambda^+(G) \bigm| d_{s,i_s}(\lambda) < k_0 \text{ for all } 1 \leqslant i_s < n_s, \text{ all } 1 \leqslant s \leqslant r\bigr\}.$$

Clearly, $\#(X/Z \cap X) = k_0^{\sum_{s=1}^r (n_s-1)}$ is finite. If $\lambda \in \Lambda^+(G) \smallsetminus X$, then $\lambda = \lambda' \lambda_{s,i}^{d_{s,i}(\lambda)}$, for some $\lambda' \in \Lambda^+(G)$, some $1 \leqslant s \leqslant r$ and $1 \leqslant i < n_s$ with $d_{s,i}(\lambda) \geqslant k_0$. By (3.1) we have

$$\phi_v(\lambda) = \pi(c_{\lambda'})\phi_v(\lambda_{s,i}^{d_{s,i}(\lambda)}) = 0.$$

It follows that $\operatorname{Supp}\phi_v \subseteq X$ is finite modulo Z.

"(b) $\Longrightarrow$ (c)": Let $\xi \in \widetilde{V}$ and $v \in V$, both non-zero. Let $H \subseteq G$ be a compact open subgroup such that $\xi \in \widetilde{V}^H$. Then $\xi = \xi \circ \pi(e_H)$ and hence

$$\bigl\langle \xi, f_{H,v}(g)\bigr\rangle = \bigl\langle \xi, \pi(e_H)\pi(g^{-1})v\bigr\rangle = \bigl\langle \xi, \pi(g^{-1})v\bigr\rangle = m_{\xi,v}(g)$$

for all $g \in G$. We deduce $\operatorname{Supp} m_{\xi,v} \subseteq \operatorname{Supp} f_{H,v}$.

"(c) $\Longrightarrow$ (d)": Let $\xi \in \widetilde{V}$ and $v \in V$, both non-zero. By assumption, the matrix coefficient $m_{\xi,v}$ has compact support modulo Z. Fix a compact open subgroup $H \subseteq G^0$ such that $\xi \in \widetilde{V}^H$ and $v \in V^H$. Then, there exist $g_1, \ldots, g_d \in G$ such that $\operatorname{Supp} m_{\xi,v} = \bigsqcup_{i=1}^d Hg_iZH$. Without loss of generality, we may assume that $g_i \in G^0$ provided $g_iZ \cap G^0 \neq \varnothing$; this implies $Hg_iZH \cap G^0 \subseteq Hg_i(Z \cap G^0)H$. Then

$$\operatorname{Supp} m_{\xi,v} \cap G^0 \subseteq \bigsqcup_{i=1}^d Hg_i(Z \cap G^0)H$$

which shows that $m_{\xi,v} \colon G^0 \to \mathbb{C}$ has compact support. By Theorem 11.4 it follows that $(V, \pi_{|G^0})$ is compact.

"(d) $\Longrightarrow$ (a)": Let $P = MN$ be a proper parabolic subgroup of G and fix $\lambda \in \Lambda^{++}(M, G) \cap G^0$ (*Exercise:* Check that such λ exists!). We have to show $V = V(N)$. Let $v \in V$ and choose $m \geqslant 0$ such that $v \in V^{K_m}$. By assumption, the function

$$\begin{aligned} f_{K_m,v} \colon G^0 &\longrightarrow V, \\ g &\longmapsto \pi(e_{K_m})\pi(g^{-1})v = \pi(c_{g^{-1}})v \end{aligned}$$

has compact support, where $c_{g^{-1}} \coloneqq e_{K_m g^{-1} K_m} \in \mathcal{H}(G, K_m)$ is the element from Lemma 13.1. In particular, $f_{K_m,v}(\lambda^{-l}) = \pi(c_{\lambda^l})v = 0$ for $l \gg 0$. By Proposition 13.3 we have $v \in V^{K_m} \cap \bigcup_{l \geqslant 0} \operatorname{Ker} \pi(c_{\lambda^l}) = V^{K_m} \cap V(N)$. Hence $v \in V(N)$. □

Theorem 15.4. *Every irreducible smooth representation of G is admissible.*

Proof. Let $(V, \pi) \in \operatorname{Rep}(G)$ be irreducible. By Lemma 15.2 there exists a parabolic subgroup $P = MN$ and an irreducible cuspidal representation $(W, \tau) \in \operatorname{Rep}(M)$ such that $(V, \pi) \subseteq \boldsymbol{i}_P^G(W, \tau)$. We first argue that (W, τ) is admissible. Let $g_1, \ldots, g_l \in M$ such that $M = \bigsqcup_{i=1}^{l} Z(M)M^0 g_i$. Since each $w \in W \smallsetminus \{0\}$ generates W as a M-representation, we deduce that $\{\tau(g_1)w, \ldots, \tau(g_l)w\}$ generates W as a $Z(M)M^0$-representation. By Corollary 12.5, the center $Z(M)$ acts on W through a character. Hence, $\{\tau(g_i)w\}_{1 \leqslant i \leqslant l}$ generates W as a M^0-representation. Proposition 11.5 combined with Theorem 15.3 implies that $(W, \tau_{|M^0})$ is admissible. But then (W, τ) is an admissible M-representation. Now, $\boldsymbol{i}_P^G(W, \tau)$ is admissible by Theorem 14.3(d). Thus, also the subrepresentation (V, π) is admissible. □

Exercise. Let $(V, \pi) \in \operatorname{Rep}(G)$. Show that (V, π) is (cuspidal and) irreducible if and only if $(\widetilde{V}, \widetilde{\pi})$ is (cuspidal and) irreducible.

It turns out that one can prove a stronger version of Theorem 15.4.

Burnside's Theorem 15.5. *Let R be an associative $\mathbb{C}$-algebra and W a finite dimensional simple R-module. The action map $R \twoheadrightarrow \operatorname{End}_{\mathbb{C}}(W)$ is surjective.*

Proof. Note that $\operatorname{End}_R(W) \cong \mathbb{C}$ by Schur's Lemma 8.6. Let $w_1, \ldots, w_d$ be a $\mathbb{C}$-basis of W. For all $v_1, \ldots, v_d \in W$, Jacobson's Density Theorem 10.9 provides $r \in R$ such that $rw_i = v_i$ for all $1 \leqslant i \leqslant d$. This proves the claim. □

Theorem 15.6 (Uniform Admissibility)**.** *Let $H \subseteq G$ be a compact open subgroup of G. There exists a constant $c = c(G, H) > 0$ such that $\dim V^H \leqslant c$ for every irreducible $(V, \pi) \in \operatorname{Rep}(G)$.*

Proof. Let $(V, \pi) \in \operatorname{Rep}(G)$ be irreducible and hence admissible by Theorem 15.4. Let $m \geqslant 1$ such that $K_m \subseteq H$. Since $V^H \subseteq V^{K_m}$, we may assume from the start that $H = K_m$. By Theorem 7.9(a), V^H is a simple $\mathcal{H}(G, H)$-module. By Burnside's Theorem 15.5, it follows that the action map $\mathcal{H}(G, H) \twoheadrightarrow \operatorname{End}_{\mathbb{C}}(V^H)$ is surjective. Recall from Theorem 13.2 that

$$\mathcal{H}(G, H) = \mathcal{H}(K, H) C \, \mathcal{H}(K, H),$$

where $C \subseteq \mathcal{H}(G, H)$ is the commutative subalgebra spanned by the $c_\lambda = e_{H\lambda H}$ for $\lambda \in \Lambda^+(G)$. Recall also from the proof of Theorem 13.2 that C is generated by $l \coloneqq \sum_{s=1}^{r}(n_s + 1)$ elements. By Lemma 15.7 we have $\dim \pi(C) \leqslant (\dim V^H)^{2 - 2^{1-l}}$. We now estimate

$$\begin{aligned}(\dim V^H)^2 = \dim \operatorname{End}_{\mathbb{C}}(V^H) = \dim \pi(\mathcal{H}(G, H)) &\leqslant (\dim \mathcal{H}(K, H))^2 \cdot \dim \pi(C) \\ &\leqslant \big(\dim \mathcal{H}(K, H)\big)^2 \cdot (\dim V^H)^{2 - 2^{1-l}}.\end{aligned}$$

Hence, rearranging gives $\dim V^H \leqslant c(G, H) \coloneqq (\dim \mathcal{H}(K, H))^{2^l}$. □

Lemma 15.7. *Let V be a $\mathbb{C}$-vector space of dimension d and let $R \subseteq \operatorname{End}_{\mathbb{C}}(V)$ be a commutative subalgebra generated (as a $\mathbb{C}$-algebra) by elements $a_1, \dots, a_l \in R$. Then*

$$\dim R \leqslant f_l(d) \coloneqq d^{2-2^{1-l}}.$$

Proof. Step 0: We have $f_l(a+b) \geqslant f_l(a) + f_l(b)$ for all $a, b \in \mathbb{R}_{\geqslant 0}$. Note that for each $x \geqslant 0$ we have $f_l''(x) = (2-2^{1-l})(1-2^{1-l})x^{-2^{1-l}} \geqslant 0$. Hence, the function $x \mapsto f_l'(x)$ is monotonically increasing, that is, $f_l'(a+b) \geqslant f_l'(b)$ for all $a, b \in \mathbb{R}_{\geqslant 0}$. For each fixed $a \geqslant 0$ we deduce $f_l(a+b) - f_l(a) = \int_0^b f_l'(a+x)\,\mathrm{d}x \geqslant \int_0^b f_l'(x)\,\mathrm{d}x = f(b)$ for all $b \geqslant 0$. This proves the claim.

Step 1: We reduce to the case where each a_i is nilpotent. As R is commutative, all generalized eigenspaces of V are R-invariant. By the Jordan decomposition and induction on l, we find a decomposition $V = V_1 \oplus \dots \oplus V_r$ into R-invariant subspaces such that for all $1 \leqslant i \leqslant l$ and $1 \leqslant j \leqslant r$ there exists $\lambda_{ij} \in \mathbb{C}$ such that $(a_i)_{|V_j} - \lambda_{ij} \operatorname{id}_{V_j}$ is nilpotent. Denoting R_j the image of R in $\operatorname{End}_{\mathbb{C}}(V_j)$, we observe $R \subseteq \prod_{i=1}^r R_j$. Put $\dim V_j = d_j$ so that $d = d_1 + \dots + d_r$. By Step 0 we have

$$f_l(d) = f_l(d_1 + \dots + d_r) \geqslant f_l(d_1) + \dots + f_l(d_r).$$

We thus reduce to showing $\dim R_j \leqslant f_l(d_j)$ for all $1 \leqslant j \leqslant r$. Since $\{(a_i)_{|V_j} - \lambda_{ij} \operatorname{id}_{V_j}\}_i$ generates R_j, we may assume from the start that $a_1, \dots, a_l$ are nilpotent.

Step 2: Denote $\phi_l(d)$ the largest possible dimension of a commutative subalgebra $R \subseteq \operatorname{End}_{\mathbb{C}}(V)$ generated by nilpotent elements $a_1, \dots, a_l$. We claim

$$\phi_l(d) \leqslant \phi_l\big(\lfloor d - \phi_l(d)/d \rfloor\big) + \phi_{l-1}(d), \quad \text{for all } d \geqslant 0,\ l \geqslant 1. \tag{3.9}$$

Let $\mathfrak{a}$ be the ideal generated by $a_1, \dots, a_l$, and put $V_j \coloneqq \mathfrak{a}^j V$. We thus have a chain of subspaces

$$\{0\} = V_d \subseteq V_{d-1} \subseteq \dots \subseteq V_1 \subseteq V_0 = V;$$

here, $V_d = \mathfrak{a}^d V = \{0\}$ follows from the fact that, if $\mathfrak{a}^j V = \mathfrak{a}^{j+1} V$ for some j, then $\mathfrak{a}^j V = \mathfrak{a}^{j+i} V$ for all $i \geqslant 0$ and hence $\mathfrak{a}^j V = \{0\}$, because $\mathfrak{a}$ is a nilpotent ideal. Let W be a complement of V_1 in V of dimension m. Note that $\mathfrak{a}^j W + \mathfrak{a}^{j+1} V = \mathfrak{a}^j (W + V_1) = \mathfrak{a}^j V = V_j$, that is, $\mathfrak{a}^j W$ generates V_j modulo V_{j+1}. It follows that $RW = V$. But this means that the composite $R \hookrightarrow \operatorname{End}_{\mathbb{C}}(V) \to \operatorname{Hom}_{\mathbb{C}}(W, V)$ is injective (here, the second map is given by restriction). We deduce $\dim R \leqslant md$ and hence $m \geqslant \phi_l(d)/d$. Let $R' \subseteq R$ be the subalgebra generated by $a_2, \dots, a_l$ and let $\mathfrak{b} = a_1 R$. Then $R = R' + \mathfrak{b}$ and $\dim R' \leqslant \phi_{l-1}(d)$. The composite $V \xrightarrow{a_1} a_1 V \subseteq V_1 \subseteq V$ induces a commutative diagram

$$\begin{array}{ccccc} \operatorname{End}_{\mathbb{C}}(V) & & \xrightarrow{\quad a_1 \quad} & & \\ \downarrow & & & & \searrow \\ \operatorname{Hom}_{\mathbb{C}}(V_1, V) & \longrightarrow & \operatorname{Hom}_{\mathbb{C}}(a_1 V, V) & \xrightarrow[f \mapsto f \circ a_1]{} & \operatorname{End}_{\mathbb{C}}(V). \end{array}$$

The image of R under the diagonal map is $\mathfrak{b}$. Hence, the image R'' of R under the vertical arrow maps surjectively onto $\mathfrak{b}$. Observe that R'' is in fact contained in $\operatorname{End}_{\mathbb{C}}(V_1)$. As ϕ_l is monotonically increasing, we deduce

$$\dim \mathfrak{b} \leqslant \dim R'' \leqslant \phi_l(\dim V_1) = \phi_l(d-m) \leqslant \phi_l\big(\lfloor d - \phi_l(d)/d \rfloor\big).$$

Together, we obtain $\dim R \leqslant \dim \mathfrak{b} + \dim R' \leqslant \phi_l(\lfloor d - \phi_l(d)/d \rfloor) + \phi_{l-1}(d)$ proving the claim.

Step 3: We claim that

$$f_l\big(\lfloor d - f_l(d)/d \rfloor\big) + f_{l-1}(d) \leqslant f_l(d).$$

Once this is established, we obtain $\phi_l(d) \leqslant f_l(d)$ from the claim and (3.9) by induction on d and l. This then finishes the proof of the lemma.

Put $\varepsilon := 2^{1-l}$ and note $0 < \varepsilon \leqslant 1$. Let $d \geqslant 1$, so that $(1-d^{-\varepsilon})^{2-\varepsilon} \leqslant 1-d^{-\varepsilon}$. Since $2^{1-(l-1)} = 2\varepsilon$, we compute

$$\begin{aligned} f_l\big(\lfloor d - f_l(d)/d \rfloor\big) + f_{l-1}(d) &\leqslant \big(d - d^{1-\varepsilon}\big)^{2-\varepsilon} + d^{2-2\varepsilon} \\ &= d^{2-\varepsilon} \cdot \big(1 - d^{-\varepsilon}\big)^{2-\varepsilon} + d^{2-2\varepsilon} \\ &= d^{2-\varepsilon} \cdot \Big(\big(1 - d^{-\varepsilon}\big)^{2-\varepsilon} + d^{-\varepsilon}\Big) \\ &\leqslant d^{2-\varepsilon} \cdot \big(1 - d^{-\varepsilon} + d^{\varepsilon}\big) \\ &= f_l(d). \end{aligned}$$

This finishes the proof. □

Variant 15.8. *Let $H \subseteq G^0$ be a compact open subgroup. Then $\dim W^H \leqslant c(G,H)$ for every irreducible $(W,\tau) \in \operatorname{Rep}(G^0)$, where $c(G,H)$ is the constant from Theorem 15.6.*

Proof. Let $(W,\tau) \in \operatorname{Rep}(G^0)$ be an irreducible representation. It is clear that $\operatorname{ind}_{G^0}^G \tau$ is finitely generated and hence admits a quotient $\operatorname{ind}_{G^0}^G \tau \twoheadrightarrow \sigma$, where $(E,\sigma) \in \operatorname{Rep}(G)$ is an irreducible representation. By Frobenius reciprocity 9.9, we have a natural bijection

$$\operatorname{Hom}_G\big(\operatorname{ind}_{G^0}^G \tau, \sigma\big) \cong \operatorname{Hom}_{G^0}\big(\tau, \sigma_{|G^0}\big).$$

Hence, we obtain a non-zero map $(W,\tau) \to (E, \sigma_{|G^0})$, which is injective as τ is irreducible. For each compact open subgroup $H \subseteq G^0$, we deduce $\dim W^H \leqslant \dim E^H \leqslant c(G,H)$. □

We finish with some consequences of Variant 15.8.

Proposition 15.9. *Fix a compact open subgroup $H \subseteq G$. There exists a compact open subset $\Omega = \Omega(G^0, H) \subseteq G^0$ such that for all irreducible compact $(W,\tau) \in \operatorname{Rep}(G^0)$ and all $w \in W^H$, we have $\operatorname{Supp} f_{H,w} \subseteq \Omega$.*

Proof. Let $(W,\tau) \in \operatorname{Rep}(G^0)$ be an irreducible compact representation and let $w \in W^H$. Let $m \geqslant 1$ such that $K_m \subseteq H$. As $\operatorname{Supp} f_{H,w} \subseteq \operatorname{Supp} f_{K_m,w}$, we may assume from the start that $H = K_m$. Put $\Lambda^+(G^0) := \Lambda^+(G) \cap G^0$ and consider the function

$$\begin{aligned} \phi_w \colon \Lambda^+(G^0) &\longrightarrow W^H, \\ \lambda &\longmapsto \tau(c_\lambda)w = f_{H,w}(\lambda^{-1}), \end{aligned}$$

where $c_\lambda = e_{H\lambda H}$ in $\mathcal{H}(G^0, H)$. As already observed in the proof of "(a) $\Longrightarrow$ (b)" in Theorem 15.3, we have $(\operatorname{Supp} f_{H,w})^{-1} \subseteq \bigcup_{k \in K/H} K \operatorname{Supp} \phi_{\tau(k)w} K$. Hence, it suffices to find a finite subset $\Omega' \subseteq \Lambda^+(G^0)$ such that $\operatorname{Supp} \phi_w \subseteq \Omega'$ for all $w \in W^H$ and all irreducible compact $(W,\tau) \in \operatorname{Rep}(G^0)$, because then $\Omega = K(\Omega')^{-1}K$ has the desired properties.

Claim. The monoid $\Lambda^+(G^0)$ is finitely generated.

Proof. Since $\Lambda^+(G^0) = \prod_{i=1}^r \Lambda^+(\mathrm{GL}_{n_i}(F)^0)$, we may assume without loss of generality that $G = \mathrm{GL}_n(F)$. Identifying $\mathrm{diag}(\varpi^{m_1}, \dots, \varpi^{m_n})$ with $(m_1, \dots, m_n) \in \mathbb{Z}^n \subseteq \mathbb{Q}^n$, we have to show that the additive monoid $M \coloneqq \{(m_1, \dots, m_n) \in \mathbb{Z}^n \mid m_1 \geqslant \dots \geqslant m_n \text{ and } \sum_{i=1}^n m_i = 0\}$ is finitely generated. For each $1 \leqslant i \leqslant n-1$, we denote $\lambda_i = (\lambda_{i,1}, \dots, \lambda_{i,n}) \in \mathbb{Q}^n$ the unique element satisfying $\sum_{j=1}^n \lambda_{i,j} = 0$ and $\lambda_{i,j} - \lambda_{i,j+1} = \delta_{ij}$ for all $1 \leqslant j \leqslant n-1$. More explicitly, we put

$$\lambda_i \coloneqq \frac{1}{n}\big(\underbrace{n-i, \dots, n-i}_{i \text{ times}}, \underbrace{-i, \dots, -i}_{(n-i)\text{-times}}\big).$$

Consider the finite set $X \coloneqq \left\{\sum_{i=1}^{n-1} a_i\lambda_i \,\middle|\, a_1, \dots, a_{n-1} \in \{0, 1, \dots, n-1\}\right\} \cap \mathbb{Z}^n$. We claim that a generating set for M is then given by $X \cup \{n\lambda_1, \dots, n\lambda_{n-1}\}$. Indeed, it is clear that this set is contained in M. Note that every $x = (x_1, \dots, x_n) \in M$ is uniquely determined by the sequence of differences $x_1 - x_2, x_2 - x_3, \dots, x_{n-1} - x_n$, because of $x_1 + x_2 + \dots + x_n = 0$. But this means $x = \sum_{i=1}^{n-1}(x_i - x_{i+1}) \cdot \lambda_i$. Writing $x_i - x_{i+1} = a_i n + b_i$ with $a_i \in \mathbb{Z}_{\geqslant 0}$ and $0 \leqslant b_i < n$, we see that $x = \sum_{i=1}^{n-1} b_i\lambda_i + \sum_{i=1}^{n-1} a_i \cdot n\lambda_i$ can be expressed (non-uniquely) as a sum of elements in $X \cup \{n\lambda_1, \dots, n\lambda_{n-1}\}$. □

We fix a family of generators $\nu_1, \dots, \nu_l$ of $\Lambda^+(G^0)$ and want to show that

$$\Omega' \coloneqq \left\{\prod_{i=1}^{l} \nu_i^{d_i} \,\middle|\, 0 \leqslant d_1, \dots, d_l \leqslant c(G, H)\right\}$$

has the desired properties. We will deduce this from the following claim:

Claim. Let $\lambda \in \Lambda^+(G^0)$ with $\lambda \neq 1$. Let $n_0 \in \mathbb{Z}_{\geqslant 0}$ such that $\phi_w(\lambda^{n_0}) \neq 0$. Then $\{\phi_w(\lambda^j)\}_{j=1}^{n_0} \subseteq W^H$ is linearly independent. In particular, $n_0 \leqslant c(G, H)$.

Proof of the claim. Note that ϕ_w has finite support, since (W, τ) is compact. Let $N \in \mathbb{Z}_{>0}$ be the smallest integer with $\phi_w(\lambda^N) = 0$. Using the relations (3.1) (that is, $c_{\lambda\lambda'} = c_\lambda * c_{\lambda'}$ for all $\lambda, \lambda' \in \Lambda^+(G)$), we see that $\phi_w(\lambda^{N+i}) = \tau(c_{\lambda^i})\phi_w(\lambda^N) = 0$ for all $i \geqslant 0$. In particular, we have $\phi_w(\lambda^j) \neq 0$ for all $1 \leqslant j \leqslant n_0$. Increasing n_0 if necessary, we may assume $n_0 = N - 1$. Let $a_1, \dots, a_{n_0} \in \mathbb{C}$ such that $x \coloneqq \sum_{j=1}^{n_0} a_j\phi_w(\lambda^j) = 0$. Then

$$0 = \tau(c_{\lambda^{n_0-i}})x = \sum_{j=1}^{i} a_j\phi_w(\lambda^{n_0-i+j})$$

for all $i = 1, \dots, n_0$. We inductively deduce $a_1 = a_2 = \dots = a_{n_0} = 0$. The last assertion follows from Variant 15.8. □

Let now $\lambda \in \Lambda^+(G^0) \smallsetminus \Omega'$ and write $\lambda = \lambda'\nu_i^{d_i}$ for some $\lambda' \in \Lambda^+(G^0)$ and some i with $d_i > c(G, H)$. The claim applied to ν_i, together with (3.1), shows

$$\phi_w(\lambda) = \tau(c_{\lambda'})\phi_w\big(\nu_i^{d_i}\big) = 0.$$

Hence, $\operatorname{Supp}\phi_w \subseteq \Omega'$. This finishes the proof. □

Corollary 15.10. *Let $H \subseteq G$ be a compact open subgroup. Then G^0 has only finitely many isomorphism classes of irreducible compact representations (W, τ) with $W^H \neq \{0\}$.*

Proof. Let $(W_1, \tau_1), \dots, (W_l, \tau_l)$ be pairwise non-isomorphic irreducible compact G^0-representations. Fix non-zero vectors $w_i \in W_i^H$ and $\xi_i \in \widetilde{W}_i^H$ for all $1 \leqslant i \leqslant l$. Since $\xi_i \circ f_{H,w_i} = m_{\xi_i,w_i}$, it follows that $\operatorname{Supp} m_{\xi_i,w_i} \subseteq \Omega(G^0, H)$ for all i, where $\Omega(G^0, H) \subseteq G^0$ denotes the compact open subset from Proposition 15.9. We may assume that $\Omega(G^0, H) = H\Omega(G^0, H)H$. Then $H\backslash\Omega(G^0, H)/H$ has finite cardinality, say, L, and hence the space $C_c^\infty(\Omega(G^0, H), H)$ of H-biinvariant functions has dimension L. The following claim shows $l \leqslant L$, which then finishes the proof.

Claim. The matrix coefficients m_{ξ_i,w_i}, for $1 \leqslant i \leqslant l$, are linearly independent in $C_c^\infty(\Omega(G^0, H), H)$.

Proof of the claim. Let $a_1, \dots, a_l \in \mathbb{C}$ such that $x \coloneqq \sum_{i=1}^{l} a_i m_{\xi_i,w_i} = 0$. By Proposition 11.10 we have $\tau_j \circ m_{\xi_i,w_i} = 0$ for $j \neq i$, and $\tau_i \circ m_{\xi_i,w_i} = d(\tau_i)^{-1} \cdot w_i \otimes \xi_i$, where $d(\tau_i)$ denotes the formal degree of τ_i. Hence, for each $1 \leqslant j \leqslant l$ we have $0 = \tau_j \circ x = d(\tau_j)^{-1} a_j w_j \otimes \xi_j$. We deduce $a_1 = \cdots = a_l = 0$. □

□

§16. Interlude: Decomposition of Categories

For this section only, let G be a locally profinite group.

Definition 16.1. (a) We denote $\mathbf{Irr}(G)$ the set of isomorphism classes of irreducible smooth G-representations. Given an irreducible G-representation (V, π), we denote $[(V, \pi)]$ the isomorphism class of (V, π). By abuse of notation we usually write $(V, \pi) \in \mathbf{Irr}(G)$.

(b) Let $(V, \pi) \in \operatorname{Rep}(G)$. We denote $\operatorname{JH}(V)$ (or $\operatorname{JH}(\pi)$) the set of (isomorphism classes of) irreducible subquotients (also called *Jordan–Hölder factors* of (V, π)).

(c) We say (V, π) has *finite length* if there exists a finite filtration $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_l = V$ of G-invariant subspaces such that V_i/V_{i-1} is irreducible for all $1 \leqslant i \leqslant l$. The integer $\ell(V) \coloneqq l$ is called the *length* of V.

Lemma 16.2. *Let $(V, \pi) \in \operatorname{Rep}(G)$, and let $\{0\} = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_l = V$ be a finite filtration of G-invariant subspaces such that V_i/V_{i-1} is irreducible for all $1 \leqslant i \leqslant l$.*

(a) One has $\operatorname{JH}(V) = \{[V_i/V_{i-1}] \mid 1 \leqslant i \leqslant l\}$. In particular, $\operatorname{JH}(V)$ is finite.

(b) Suppose G is countable at infinity. If $(W, \sigma) \in \operatorname{Rep}(G)$ has finite length, then $\operatorname{Hom}_G(V, W)$ is finite dimensional.

Proof. In (a), the relation "$\supseteq$" is trivial, so we only need to prove "$\subseteq$". Let $W' \subseteq W \subseteq V$ be G-invariant subspaces such that W/W' is irreducible. Let i be the unique index such that $W \cap V_{i-1} \subseteq W'$ and $W \cap V_i \nsubseteq W'$. We then have $W \cap V_{i-1} = W' \cap V_{i-1}$ and $W' \cap V_i \subsetneq W \cap V_i$. We deduce $W \cap V_{i-1} = W' \cap V_{i-1} \subseteq W' \cap V_i \subsetneq W \cap V_i$. We obtain non-zero maps

$$\frac{W}{W'} \longleftarrow \frac{W \cap V_i}{W' \cap V_i} \twoheadleftarrow \frac{W \cap V_i}{W' \cap V_{i-1}} = \frac{W \cap V_i}{W \cap V_{i-1}} \hookrightarrow \frac{V_i}{V_{i-1}}.$$

As W/W' and V_i/V_{i-1} are irreducible, all maps are isomorphisms. This shows (a).

We prove (b) by induction on $l + \ell(W)$. If $l + \ell(W) \leqslant 2$, then $\dim \operatorname{Hom}_G(V, W) \leqslant 1$ by Schur's Lemma 11.6. Let now $l + \ell(W) > 2$. If $l > 1$, then we have an exact sequence $\{0\} \to \operatorname{Hom}_G(V_l/V_{l-1}, W) \to \operatorname{Hom}_G(V, W) \to \operatorname{Hom}_G(V_{l-1}, W)$. The induction hypothesis implies $\dim \operatorname{Hom}_G(V, W) \leqslant \dim \operatorname{Hom}_G(V_l/V_{l-1}, W) + \dim \operatorname{Hom}_G(V_{l-1}, W) < \infty$. Similarly, if $\ell(W) > 1$, let $\{0\} \neq W' \subsetneq W$ be a proper G-invariant subspace so that $0 < \ell(W'), \ell(W/W') < \ell(W)$. We then obtain an exact sequence $\{0\} \to \operatorname{Hom}_G(V, W') \to \operatorname{Hom}_G(V, W) \to \operatorname{Hom}_G(V, W/W')$, and the induction hypothesis implies $\dim \operatorname{Hom}_G(V, W) \leqslant \dim \operatorname{Hom}_G(V, W') + \dim \operatorname{Hom}_G(V, W/W') < \infty$. □

Lemma 16.3. *Let $(V, \pi) \in \operatorname{Rep}(G)$.*

(a) If $W \subseteq V$ is a G-invariant subspace, then

$$\operatorname{JH}(V) = \operatorname{JH}(W) \cup \operatorname{JH}(V/W).$$

(b) One has $\operatorname{JH}(V) = \varnothing$ if and only if $V = \{0\}$.

(c) Let $\{W_i\}_{i \in I}$ be a family of G-invariant subspaces of V. Then

$$\operatorname{JH}\Big(\sum_{i \in I} W_i\Big) = \bigcup_{i \in I} \operatorname{JH}(W_i).$$

Proof. We prove (a). The inclusion $\operatorname{JH}(W) \subseteq \operatorname{JH}(V)$ is obvious, and $\operatorname{JH}(V/W) \subseteq \operatorname{JH}(V)$ follows from the third isomorphism theorem. Conversely, let $V'' \subseteq V' \subseteq V$ be G-invariant subspaces such that V'/V'' is irreducible (and hence defines an element of $\operatorname{JH}(V)$). If $V' \cap W \not\subseteq V''$, then the projection map $V' \cap W \to V'/V''$ is non-zero and hence surjective, since V'/V'' is irreducible. It follows that $V'/V'' \in \operatorname{JH}(W)$. If $V' \cap W \subseteq V''$, then

$$\begin{array}{ccc} V'/V'' \cap W & \hookrightarrow & V/W \\ \big\downarrow\!\!\!\downarrow & & \\ V'/V'' & & \end{array}$$

shows $V'/V'' \in \operatorname{JH}(V/W)$.

For part (b), it it obvious that $V = \{0\}$ implies $\operatorname{JH}(V) = \varnothing$. If $V \neq \{0\}$, let $V' \subseteq V$ be the G-invariant subspace generated by a non-zero vector $v \in V$. By Zorn's Lemma there exists a maximal G-invariant subspace $V'' \subseteq V'$ with $v \notin V''$, so that $V'/V'' \in \operatorname{JH}(V)$.

We prove (c). The inclusion "$\supseteq$" follows from (a). Let now $V'' \subseteq V'$ be G-invariant subspaces of $\sum_{i \in I} W_i$ such that V'/V'' is irreducible. Let $v \in V' \smallsetminus V''$. There exist $i_1, \ldots, i_n \in I$ such that $v \in \sum_{j=1}^n W_{i_j}$. Hence the G-invariant subspace $X = \mathbb{C}[G]v$ generated by v is contained in $\sum_{j=1}^n W_{i_j}$. As the map $X/X \cap V'' \hookrightarrow V'/V''$ is non-zero and V'/V'' is irreducible, it is an isomorphism. Hence $V'/V'' \cong X/X \cap V'' \in \operatorname{JH}(\sum_{j=1}^n W_{i_j})$. Define $Y_k \coloneqq \sum_{j=1}^k W_{i_j}$ for each $1 \leqslant k \leqslant n$ and $Y_0 \coloneqq \{0\}$. Then $W_{i_k} \twoheadrightarrow Y_k/Y_{k-1}$ is surjective so that $\operatorname{JH}(Y_k/Y_{k-1}) \subseteq \operatorname{JH}(W_{i_k})$. Applying (a) repeatedly, we obtain

$$V'/V'' \in \operatorname{JH}(Y_n) = \operatorname{JH}(Y_n/Y_{n-1}) \cup \operatorname{JH}(Y_{n-1}) = \cdots = \bigcup_{k=1}^n \operatorname{JH}(Y_k/Y_{k-1}) \subseteq \bigcup_{k=1}^n \operatorname{JH}(W_{i_k}). \qquad \square$$

Definition 16.4. (a) Let $\{\mathscr{C}_i\}_{i\in I}$ be a family of full subcategories of $\mathrm{Rep}(G)$. We write

$$\mathrm{Rep}(G) = \prod_{i\in I} \mathscr{C}_i \tag{3.10}$$

if every $(V,\pi) \in \mathrm{Rep}(G)$ decomposes as $V = \bigoplus_{i\in I} V_i$, where $V_i \in \mathscr{C}_i$ for $i \in I$, and for all $V_i \in \mathscr{C}_i$ and $V_j \in \mathscr{C}_j$ with $i \neq j$, we have $\mathrm{Hom}_G(V_i, V_j) = \{0\}$.

We denote $\mathbf{Irr}(\mathscr{C}_i)$ the set of isomorphism classes of irreducible smooth G-representations (V,π) that lie in $\mathscr{C}_i$. Note that (3.10) implies $\mathbf{Irr}(G) = \bigsqcup_{i\in I} \mathbf{Irr}(\mathscr{C}_i)$.

(b) Let $S \subseteq \mathbf{Irr}(G)$.

- We denote $\mathrm{Rep}(G)_S$ the full subcategory of $\mathrm{Rep}(G)$ of all (V,π) with $\mathrm{JH}(V) \subseteq S$. Hence, $\mathbf{Irr}(\mathrm{Rep}(G)_S) = S$. Note that $\mathrm{Rep}(G)_S$ is closed under the formation of subquotients, extensions, and direct sums by Lemma 16.3.
- If $(V,\pi) \in \mathrm{Rep}(G)$ we denote by V_S the sum of all G-invariant subspaces of V which lie in $\mathrm{Rep}(G)_S$. Note that V_S is the largest subrepresentation of V with $V_S \in \mathrm{Rep}(G)_S$.

Lemma 16.5. *Let $S, S' \subseteq \mathbf{Irr}(G)$ with $S \cap S' = \varnothing$.*

(a) Let $(V,\pi) \in \mathrm{Rep}(G)$. Then $V_S \cap V_{S'} = \{0\}$ and hence $V_S \oplus V_{S'} \subseteq V$.

(b) For all $(V,\pi) \in \mathrm{Rep}(G)_S$ and $(V',\pi') \in \mathrm{Rep}(G)_{S'}$ we have $\mathrm{Hom}_G(V, V') = \{0\}$.

Proof. We prove (a). Since $\mathrm{JH}(V_S \cap V_{S'}) \subseteq \mathrm{JH}(V_S) \cap \mathrm{JH}(V_{S'}) \subseteq S \cap S' = \varnothing$, we deduce $V_S \cap V_{S'} = \{0\}$ from Lemma 16.3(b).

For part (b), observe first that $V'_{S'} = V'$ and hence $V'_S = \{0\}$ by (a). For each $f \in \mathrm{Hom}_G(V, V')$, the image $\mathrm{Im}(f)$ is a quotient of V and hence lies in $\mathrm{Rep}(G)_S$. Therefore, we have $\mathrm{Im}(f) \subseteq V'_S = \{0\}$, which shows $f = 0$. ☐

Definition 16.6. Let $\mathbf{Irr}(G) = \bigsqcup_{\alpha\in A} S_\alpha$ be a partition. We say that $\{S_\alpha\}_\alpha$ *splits* an object $(V,\pi) \in \mathrm{Rep}(G)$ if

$$V = \bigoplus_{\alpha\in A} V_{S_\alpha}.$$

We say $\{S_\alpha\}_\alpha$ *splits* $\mathrm{Rep}(G)$ if it splits every object of $\mathrm{Rep}(G)$, that is,

$$\mathrm{Rep}(G) = \prod_{\alpha\in A} \mathrm{Rep}(G)_{S_\alpha}.$$

Lemma 16.7. *Suppose $\mathbf{Irr}(G) = \bigsqcup_{\alpha\in A} S_\alpha$ splits $(V,\pi) \in \mathrm{Rep}(G)$. Then $\{S_\alpha\}_\alpha$ splits every subquotient of V.*

Proof. Let $W \subseteq V$ be a G-invariant subspace. It suffices to show $W = \bigoplus_\alpha W \cap V_{S_\alpha}$, because then also $V/W \cong \bigoplus_\alpha V_{S_\alpha}/W \cap V_{S_\alpha}$. Put $X := W/\bigoplus_\alpha W \cap V_{S_\alpha}$. For each α we have a surjection $W/W \cap V_{S_\alpha} \twoheadrightarrow X$ and hence

$$\mathrm{JH}(X) \subseteq \mathrm{JH}(W/W \cap V_{S_\alpha}) \subseteq \mathrm{JH}(V/V_{S_\alpha}) \subseteq \mathbf{Irr}(G) \smallsetminus S_\alpha.$$

We deduce $\mathrm{JH}(X) \subseteq \bigcap_\alpha \mathbf{Irr}(G) \smallsetminus S_\alpha = \varnothing$, and then Lemma 16.3(b) implies $X = \{0\}$. Hence, $\{S_\alpha\}_\alpha$ splits W. ☐

§17. Cuspidal Components

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \dots, n_r)$ of n. In this section we are going to relate the representations of G^0 and G and prove a first decomposition theorem for $\operatorname{Rep}(G)$.

Definition 17.1. Set $\Lambda(G) \coloneqq G/G^0 \cong \mathbb{Z}^r$. We call

$$\mathcal{X}(G) \coloneqq \operatorname{Hom}_{\mathrm{grp}}\big(\Lambda(G), \mathbb{C}^\times\big) \cong (\mathbb{C}^\times)^r$$

the set of *unramified characters* of G. An element of $\mathcal{X}(G)$ consists of a (necessarily smooth) character $\psi\colon G \to \mathbb{C}^\times$ such that $\psi(G^0) = \{1\}$. The group structure on $\mathbb{C}^\times$ turns $\mathcal{X}(G)$ into a group; concretely, for all $\phi, \psi \in \mathcal{X}(G)$, the element

$$\begin{aligned} \phi\psi\colon G &\longrightarrow \mathbb{C}^\times, \\ g &\longmapsto \phi(g)\cdot\psi(g) \end{aligned}$$

lies in $\mathcal{X}(G)$.

Remark. The group $\mathcal{X}(G)$ carries the natural structure of a $\mathbb{C}$-variety whose ring of functions is the group algebra $\mathbb{C}[\Lambda(G)]$ of $\Lambda(G)$, since

$$\mathcal{X}(G) \cong \operatorname{Hom}_{\mathrm{Alg}}\big(\mathbb{C}[\Lambda(G)], \mathbb{C}\big),$$

where "$\operatorname{Hom}_{\mathrm{Alg}}$" denotes the set of homomorphisms of $\mathbb{C}$-algebras. Since $\mathbb{C}[\Lambda(G)] \cong \mathbb{C}[t_1^{\pm 1}, \dots, t_r^{\pm 1}]$ is an integral domain, it follows that $\mathcal{X}(G)$ is in fact connected (even irreducible).

The group G acts on $\mathbf{Irr}(G^0)$ via $(g, [(W,\sigma)]) \mapsto [(W, g_*\sigma)]$, where we recall $(g_*\sigma)(\gamma) \coloneqq \sigma(g^{-1}\gamma g)$ for $\gamma \in G^0$.

Lemma 17.2. *(a) G acts on $\mathbf{Irr}(G^0)$ with finite orbits.*

(b) Let $(V,\pi) \in \operatorname{Rep}(G)$ and $(W,\sigma) \in \mathbf{Irr}(G^0)$. Denote $V(\sigma)$ the σ-isotypic component of $(V, \pi_{|G^0})$. For all $g \in G$ one has

$$\pi(g)\cdot V(\sigma) = V(g_*\sigma).$$

Proof. Let $(W,\sigma) \in \mathbf{Irr}(G^0)$ and denote $[(W,\sigma)]$ the corresponding isomorphism class. Let $z \in Z(G)$ and $\gamma \in G^0$. The $\mathbb{C}$-linear isomorphism $\sigma(\gamma)\colon \big(W, (z\gamma)_*\sigma\big) \to (W,\sigma)$ is G^0-equivariant: Indeed, for all $g \in G^0$ and $w \in W$ we compute

$$\sigma(\gamma)((z\gamma)_*\sigma)(g)w = \sigma(\gamma)\sigma(\gamma^{-1}z^{-1}gz\gamma)w = \sigma(\gamma)\sigma(\gamma^{-1}g\gamma)w = \sigma(g)\sigma(\gamma)w.$$

Hence, $(z\gamma)\cdot[(W,\sigma)] = [(W,(z\gamma)_*\sigma)] = [(W,\sigma)]$, which shows that the action of G on $\mathbf{Irr}(G^0)$ factors through the finite group $G/Z(G)G^0$. In particular, all orbits are finite.

We now prove (b). Recall that $V(\sigma)$ the image of $\operatorname{Hom}_{G^0}(\sigma, \pi_{|G^0}) \otimes W \to V$, $f \otimes w \mapsto f(w)$. Let $g \in G$. As above, we have a G^0-equivariant isomorphism $\pi(g)\colon (V, g_*\pi_{|G^0}) \to (V, \pi_{|G^0})$, $v \mapsto \pi(g)v$. Now note that the diagram

$$\begin{array}{ccccc}
\operatorname{Hom}_{G^0}\big(\sigma, \pi_{|G^0}\big) \otimes W & \twoheadrightarrow & V(\sigma) & \overset{\subseteq}{\hookrightarrow} & V \\
\| & & \vdots & & \\
\operatorname{Hom}_{G^0}\big(g_*\sigma, g_*\pi_{|G^0}\big) \otimes W & & & & \downarrow{\scriptstyle \pi(g)} \\
{\scriptstyle f \mapsto \pi(g)\circ f}\downarrow & & \downarrow & & \\
\operatorname{Hom}_{G^0}\big(g_*\sigma, \pi_{|G^0}\big) \otimes W & \twoheadrightarrow & V(g_*\sigma) & \overset{\subseteq}{\hookrightarrow} & V
\end{array}$$

commutes. It follows that the dashed arrow exists and is an isomorphism. $\square$

Proposition 17.3. *Let $(V,\pi) \in \mathbf{Irr}(G)$.*

(a) $\pi_{|G^0}$ is semisimple of finite length. Moreover, the irreducible G^0-representations contained in $\pi_{|G^0}$ form a single G-orbit.

(b) For any $(V',\pi') \in \mathbf{Irr}(G)$, the following are equivalent:

- *(i) $\pi_{|G^0} \cong \pi'_{|G^0}$;*
- *(ii) $\mathrm{JH}(\pi_{|G^0}) \cap \mathrm{JH}(\pi'_{|G^0}) \neq \varnothing$;*
- *(iii) $\pi' \cong \chi \otimes \pi$ for some $\chi \in \mathcal{X}(G)$.*

Proof. We prove (a). The subgroup $Z(G)G^0$ has finite index in G and hence $\pi_{|Z(G)G^0}$ is semisimple by Proposition 8.3. By Corollary 12.5, $Z(G)$ acts by a character on π. Hence, $\pi_{|G^0}$ is semisimple. Moreover, if $(W,\tau) \in \mathbf{Irr}(G^0)$ is contained in $(V,\pi_{|G^0})$, then so is $(\pi(g)W, \pi_{|G^0}) \cong (W, g_*\tau)$, and $V = \sum_{g \in G/Z(G)G^0} \pi(g)W$. This shows $\mathrm{JH}(\pi_{|G^0}) = \big\{[g_*\tau] \,\big|\, g \in G/Z(G)G^0\big\}$, whence (a).

We now prove (b). The implications (iii) $\Longrightarrow$ (i) $\Longrightarrow$ (ii) are obvious, so we only show (ii) $\Longrightarrow$ (iii). By (a) and Lemma 16.2, the $\mathbb{C}$-vector space $X \coloneqq \mathrm{Hom}_{G^0}(\pi_{|G^0}, \pi'_{|G^0})$ is finite dimensional. The assumption implies $X \neq \{0\}$. Define a G-representation (X,τ) via $\tau(g)f = \pi'(g) \circ f \circ \pi(g^{-1})$ for all $g \in G$ and $f \in X$. By construction, $\tau_{|G^0} \equiv 1$. Now, the abelian group $\mathbb{Z}^r \cong G/G^0$ acts on X. As $\mathbb{C}$ is algebraically closed, there exists a character $\chi \colon G/G^0 \to \mathbb{C}^\times$ and $f \in X \smallsetminus \{0\}$ such that $\tau(g)f = \chi(g) \cdot f$ for all $g \in G$. We compute

$$f\big((\chi \otimes \pi)(g)v\big) = \chi(g) \cdot f\big(\pi(g)v\big) = \big(\tau(g)f\big)\big(\pi(g)v\big) = \pi'(g)f(v)$$

for all $g \in G$ and $v \in V$. Thus, $f \colon \chi \otimes \pi \to \pi'$ is a non-zero G-equivariant map between irreducible G-representations, hence an isomorphism. $\square$

Consider now the action of $\mathcal{X}(G)$ on $\mathbf{Irr}(G)$ given by $\chi \cdot \pi \coloneqq \chi \otimes \pi$ for $\chi \in \mathcal{X}(G)$ and $(V,\pi) \in \mathbf{Irr}(G)$.

Lemma 17.4. *The stabilizer of any $(\pi, V) \in \mathbf{Irr}(G)$ is a finite subgroup of $\mathcal{X}(G)$.*

Proof. By Corollary 12.5, each $(\pi,V) \in \mathbf{Irr}(G)$ admits a central character $\chi_\pi \colon Z(G) \to \mathbb{C}^\times$. Take any $\psi \in \mathcal{X}(G)$ which stabilizes π. Then $\chi_\pi = \chi_{\psi\otimes\pi} = \psi_{|Z(G)} \cdot \chi_\pi$, so that $\psi_{|Z(G)} \equiv 1$. It follows that ψ lies in $\mathrm{Hom}_{\mathrm{grp}}(G/Z(G)G^0, \mathbb{C}^\times)$, which is finite because $G/Z(G)G^0$ is finite. $\square$

Definition 17.5. Denote $\mathbf{Irr}_{\mathrm{cusp}}(G)$ the set of (isomorphism classes of) irreducible cuspidal representations of G. By the equivalence "(a) $\Longleftrightarrow$ (d)" in Theorem 15.3, the action of $\mathcal{X}(G)$ on $\mathbf{Irr}(G)$ restricts to an action on $\mathbf{Irr}_{\mathrm{cusp}}(G)$.

An orbit of the $\mathcal{X}(G)$-action on $\mathbf{Irr}_{\mathrm{cusp}}(G)$ is called a *cuspidal component*. Observe that by Lemma 17.4, every cuspidal component D is of the form $D \cong (\mathbb{C}^\times)^r/\Gamma$, for a finite group Γ, and hence carries itself the structure of a connected $\mathbb{C}$-variety; see the following proposition.[2]

Proposition 17.6. *Let X be an affine $\mathbb{C}$-variety with coordinate ring $\mathbb{C}[X]$. Let Γ be a finite group together with a group homomorphism $\rho \colon \Gamma \to \mathrm{Aut}_{\mathbb{C}}(X)$, where $\mathrm{Aut}_{\mathbb{C}}(X)$ denotes the group of automorphisms of the $\mathbb{C}$-variety X. Then the orbit space X/Γ is an affine $\mathbb{C}$-variety with coordinate ring $\mathbb{C}[X]^\Gamma$.*

[2] In fact, one can show that there is a non-canonical isomorphism $D \cong (\mathbb{C}^\times)^r$, so D is a complex torus.

Proof. Recall that an *affine* $\mathbb{C}$*-variety* is a tuple $(Y, \mathbb{C}[Y], \mathrm{ev})$ consisting of a set Y, a finitely generated reduced commutative $\mathbb{C}$-algebra $\mathbb{C}[Y]$ and a bijection $\mathrm{ev}\colon Y \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[Y], \mathbb{C})$; we view each $f \in \mathbb{C}[Y]$ as a function on Y via $f(y) \coloneqq \mathrm{ev}(y)f$ for all $y \in Y$.[3]

A morphism $(Y_1, \mathbb{C}[Y_1], \mathrm{ev}) \to (Y_2, \mathbb{C}[Y_2], \mathrm{ev})$ of $\mathbb{C}$-varieties is a pair $(\psi, \psi^\sharp)$ consisting of a (set-theoretic) map $\psi\colon Y_1 \to Y_2$ and a $\mathbb{C}$-algebra homomorphism $\psi^\sharp\colon \mathbb{C}[Y_2] \to \mathbb{C}[Y_1]$ such that $f(\psi(y_1)) = (\psi^\sharp f)(y_1)$ for all $y_1 \in Y_1$ and $f \in \mathbb{C}[Y_2]$.

The morphism $\rho\colon \Gamma \to \mathrm{Aut}_{\mathbb{C}}(X)$ comes with a group homomorphism $\rho^\sharp\colon \Gamma \to \mathrm{Aut}_{\mathrm{Alg}}(\mathbb{C}[X])$ such that $f(\rho(g)x) = (\rho^\sharp(g^{-1})f)(x)$ for all $x \in X$, $g \in \Gamma$ and $f \in \mathbb{C}[X]$. We consider the $\mathbb{C}$-algebra

$$\mathbb{C}[X]^\Gamma \coloneqq \left\{ f \in \mathbb{C}[X] \,\middle|\, \rho^\sharp(g)f = f \text{ for all } g \in \Gamma \right\}.$$

We will prove that $\mathbb{C}[X]^\Gamma$ is finitely generated and reduced, and that there is a (necessarily unique) bijection $X/\Gamma \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X]^\Gamma, \mathbb{C})$ making the following diagram commutative:

$$\begin{array}{ccc} X & \xrightarrow{\cong} & \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X], \mathbb{C}) \\ \big\downarrow & & \big\downarrow \\ X/\Gamma & \xrightarrow[\exists! \alpha]{} & \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X]^\Gamma, \mathbb{C}). \end{array}$$

As a subring of a reduced ring, $\mathbb{C}[X]^\Gamma$ is reduced. We now prove that $\mathbb{C}[X]^\Gamma$ is a finitely generated $\mathbb{C}$-algebra. Fix $\mathbb{C}$-algebra generators $f_1, \dots, f_r \in \mathbb{C}[X]$. For each i, the monic polynomial $\chi_{f_i}(t) \coloneqq \prod_{g\in\Gamma}(t - \rho^\sharp(g) f_i) = \sum_{j=1}^{\#\Gamma} a_{ij} t^j$ lies in $\mathbb{C}[X]^\Gamma[t]$ and satisfies $\chi_{f_i}(f_i) = 0$. Let $A \subseteq \mathbb{C}[X]^\Gamma$ be the subalgebra generated by $\{a_{ij}\}_{i,j}$. It is an easy exercise to show that the finite set $\{f_1^{c_1} \cdots f_r^{c_r}\}_{0 \leqslant c_i < \#\Gamma}$ generates $\mathbb{C}[X]$ as an A-module. Since A is Noetherian, also $\mathbb{C}[X]^\Gamma$ is finitely generated over A, say, by $f_1', \dots, f_s'$. Then $\{a_{ij}\}_{i,j} \cup \{f_1', \dots, f_s'\}$ generates $\mathbb{C}[X]^\Gamma$ as a $\mathbb{C}$-algebra.

It remains to prove that the composite $X \to \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X], \mathbb{C}) \to \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X]^\Gamma, \mathbb{C})$ factors through a bijection $\alpha\colon X/\Gamma \xrightarrow{\cong} \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X]^\Gamma, \mathbb{C})$. For all $x \in X$, $g \in \Gamma$, and $f \in \mathbb{C}[X]^\Gamma$ we have $f(\rho(g)x) = (\rho^\sharp(g^{-1})f)(x) = f(x)$, that is, f is constant on Γ-orbits. This implies that there is a well-defined map α making the diagram commutative.

Let us prove that α is injective. We abbreviate $\varphi_x \coloneqq \mathrm{ev}(x)$ for $x \in X$. Let $x, y \in X$ such that $\alpha(\rho(\Gamma)x) = \alpha(\rho(\Gamma)y)$. This means $\varphi_x(f) = \varphi_y(f)$ for all $f \in \mathbb{C}[X]^\Gamma$. We have to find $g \in \Gamma$ such that $y = \rho(g)x$ or, equivalently, $\mathrm{Ker}\,\varphi_y \subseteq \mathrm{Ker}\,\varphi_{\rho(g)x}$ (for the equivalence, use that for each $\varphi \in \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[X], \mathbb{C})$ one has $\mathbb{C}[X] = \mathbb{C} \oplus \mathrm{Ker}\,\varphi$ and $\varphi_{|\mathbb{C}} = \mathrm{id}_{\mathbb{C}}$). Let $f \in \mathrm{Ker}\,\varphi_y$, and put $f' \coloneqq \prod_{g\in\Gamma} \rho^\sharp(g) f \in \mathbb{C}[X]^\Gamma$. Then

$$\prod_{g\in\Gamma} \varphi_{\rho(g)x}(f) = \varphi_x(f') = \varphi_y(f') = \prod_{g\in\Gamma} \varphi_y(\rho^\sharp(g)f) = 0 \quad \text{in } \mathbb{C}.$$

Hence, there exists $g \in \Gamma$ (depending on f) with $\varphi_{\rho(g)x}(f) = \{0\}$. So far, we have proved $\mathrm{Ker}\,\varphi_y \subseteq \bigcup_{g\in\Gamma} \mathrm{Ker}\,\varphi_{\rho(g)x}$. By the Prime Avoidance Lemma, we have $\mathrm{Ker}\,\varphi_y \subseteq \mathrm{Ker}\,\varphi_{\rho(g)x}$ for some $g \in \Gamma$. To wit, let $g_1, \dots, g_r \in \Gamma$ be a minimal set with $\mathrm{Ker}\,\varphi_y \subseteq \bigcup_{i=1}^r \mathrm{Ker}\,\varphi_{\rho(g_i)x}$, and assume for a contradiction that $r \geqslant 2$. By minimality, we have $\mathrm{Ker}\,\varphi_y \nsubseteq \bigcup_{j\neq i} \mathrm{Ker}\,\varphi_{\rho(g_j)x}$, and so we find for each

[3] Let $\mathbb{C}[t_1, \dots, t_n] \twoheadrightarrow \mathbb{C}[Y]$ be a surjective $\mathbb{C}$-algebra homomorphism with kernel $\mathfrak{a} = (a_1, \dots, a_m)$. By Hilbert's Nullstellensatz, the map $\mathrm{ev}\colon \mathbb{C}^n \to \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[t_1, \dots, t_n], \mathbb{C})$ is bijective, and under ev the inclusion $\mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[Y], \mathbb{C}) \hookrightarrow \mathrm{Hom}_{\mathrm{Alg}}(\mathbb{C}[t_1, \dots, t_n], \mathbb{C})$ identifies Y with the subset $\{y \in \mathbb{C}^n \mid a_1(y) = \dots = a_m(y) = 0\}$.

i an element $f_i \in \mathbb{C}[X]$ with $\varphi_y(f_i) = \varphi_{\rho(g_i)x}(f_i) = 0$ and $\varphi_{\rho(g_j)x}(f_i) \neq 0$ for all $j \neq i$. Consider $f \coloneqq f_1 + f_2 \cdots f_r \in \mathbb{C}[X]$. Then $\varphi_y(f) = 0$, but $\varphi_{\rho(g_1)x}(f) = \varphi_{\rho(g_1)x}(f_2) \cdots \varphi_{\rho(g_1)x}(f_r) \neq 0$ and $\varphi_{\rho(g_j)x}(f) = \varphi_{\rho(g_j)x}(f_1) \neq 0$ for all $2 \leqslant j \leqslant r$, which contradicts $\operatorname{Ker}\varphi_y \subseteq \bigcup_{i=1}^r \operatorname{Ker}\varphi_{\rho(g_i)x}$. This concludes the proof of the injectivity of α.

Finally, we show that α is surjective. Equivalently, we have to show that the restriction map $\operatorname{Hom}_{\mathrm{Alg}}(\mathbb{C}[X], \mathbb{C}) \to \operatorname{Hom}_{\mathrm{Alg}}(\mathbb{C}[X]^\Gamma, \mathbb{C})$ is surjective. Let $\varphi\colon \mathbb{C}[X]^\Gamma \to \mathbb{C}$ be a $\mathbb{C}$-algebra homomorphism. Then $\operatorname{Ker}\varphi$ is a maximal ideal of $\mathbb{C}[X]^\Gamma$. It suffices to find a maximal ideal $\mathfrak{m} \subseteq \mathbb{C}[X]$ such that $\operatorname{Ker}\varphi = \mathbb{C}[X]^\Gamma \cap \mathfrak{m}$, because then the unique $\mathbb{C}$-algebra homomorphism $\varphi'\colon \mathbb{C}[X] \to \mathbb{C}$ with $\operatorname{Ker}\varphi' = \mathfrak{m}$ extends φ.

Consider the ideal $\mathfrak{a} \coloneqq \mathbb{C}[X] \cdot \operatorname{Ker}\varphi$. We claim $\mathfrak{a} \neq \mathbb{C}[X]$. Assume for a contradiction that $1 \in \mathfrak{a}$. Then we can write $1 = \sum_{i=1}^n f_i h_i$, for certain $f_i \in \mathbb{C}[X]$ and $h_i \in \operatorname{Ker}\varphi$. But then for $\widetilde{f}_i \coloneqq \frac{1}{\#\Gamma}\sum_{g\in\Gamma} \rho^\sharp(g) f_i \in \mathbb{C}[X]^\Gamma$, we have

$$\begin{aligned}\sum_{i=1}^n \widetilde{f}_i h_i &= \sum_{i=1}^n \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\rho^\sharp(g)f_i \cdot h_i = \sum_{i=1}^n \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\rho^\sharp(g)f_i\cdot\rho^\sharp(g)h_i \\ &= \sum_{i=1}^n \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\rho^\sharp(g)(f_ih_i) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\rho^\sharp(g)\Big(\sum_{i=1}^n f_ih_i\Big) = \frac{1}{\#\Gamma}\sum_{g\in\Gamma}\rho^\sharp(g)(1) = 1.\end{aligned}$$

Hence, $1 = \sum_{i=1}^n \widetilde{f}_i h_i \in \operatorname{Ker}\varphi$, which contradicts the fact that $\operatorname{Ker}\varphi$ is a proper ideal of $\mathbb{C}[X]^\Gamma$. This shows $\mathfrak{a} \neq \mathbb{C}[X]$. By Zorn's lemma we find a maximal ideal $\mathfrak{m} \subseteq \mathbb{C}[X]$ containing $\mathfrak{a}$. By construction, we have $\operatorname{Ker}\varphi \subseteq \mathbb{C}[X]^\Gamma \cap \mathfrak{m}$. Since $\operatorname{Ker}\varphi$ is maximal, it follows that $\operatorname{Ker}\varphi = \mathbb{C}[X]^\Gamma \cap \mathfrak{m}$. □

Proposition 17.7. *Let $D \subseteq \mathbf{Irr}_{\mathrm{cusp}}(G)$ be a cuspidal component. Then D splits the category* $\operatorname{Rep}(G)$.

Proof. Put $D' \coloneqq \mathbf{Irr}(G) \smallsetminus D$ and let $(V, \pi) \in \operatorname{Rep}(G)$. We have to show $V = V_D \oplus V_{D'}$, where $V_D \in \operatorname{Rep}(G)_D$ and $V_{D'} \in \operatorname{Rep}(G)_{D'}$.

Let $(W, \sigma) \in D$. By Proposition 17.3, the restriction $\rho \coloneqq \sigma_{|G^0}$ is semisimple of finite length, only depends on the $\mathcal{X}(G)$-orbit of σ, and $\operatorname{JH}(\rho) = \{\rho_1, \dots, \rho_l\}$ forms a single G-orbit. Since σ is cuspidal, Theorem 15.3 shows that ρ is compact, hence also $\rho_1, \dots, \rho_l \in \mathbf{Irr}(G^0)$ are compact.

Put $\tau \coloneqq \pi_{|G^0}$ and recall the G^0-equivariant projections $\tau(e_{\rho_i})\colon V \to V$ from Theorem 11.12. They provide a decomposition $V = V(\rho_i) \oplus \operatorname{Ker}\tau(e_{\rho_i})$ in $\operatorname{Rep}(G^0)$, where $V(\rho_i) = \operatorname{Im}\tau(e_{\rho_i})$ is the ρ_i-isotypic component of V and $\rho_i \notin \operatorname{JH}(\operatorname{Ker}\tau(e_{\rho_i}))$. These satisfy the following properties:

(i) $\tau(e_{\rho_i}) \circ \tau(e_{\rho_j}) = 0$ for all $i \neq j$. Indeed, we have $\operatorname{JH}\big(\tau(e_{\rho_i})\operatorname{Im}\tau(e_{\rho_j})\big) \subseteq \{\rho_i\} \cap \{\rho_j\} = \varnothing$. Lemma 16.3(b) now shows $\tau(e_{\rho_i})(\operatorname{Im}\tau(e_{\rho_j})) = \{0\}$.

(ii) $\tau(g)\tau(e_{\rho_i}) = \tau(e_{g_*\rho_i})\tau(g)$ for all i and $g \in G$. We may check the equality after restriction to $V(\rho_i)$ and $\operatorname{Ker}\tau(e_{\rho_i})$ separately. Lemma 17.2(b) shows $\tau(g)V(\rho_i) = V(g_*\rho_i)$. Hence, it remains to check $\tau(g)\operatorname{Ker}\tau(e_{\rho_i}) \subseteq \operatorname{Ker}\tau(e_{g_*\rho_i})$. Note that any irreducible subquotient κ of $\tau(g)\operatorname{Ker}\tau(e_{\rho_i})$ satisfies $\kappa \ncong g_*\rho_i$. Hence, $\operatorname{JH}\big(\tau(e_{g_*\rho_i})\tau(g)\operatorname{Ker}\tau(e_{\rho_i})\big) = \varnothing$ and Lemma 16.3(b) shows $\tau(e_{g_*\rho_i})\tau(g)\operatorname{Ker}\rho(e_{\rho_i}) = \{0\}$.

By (i), we obtain a decomposition

$$V = \bigoplus_{i=1}^l V(\rho_i) \oplus V', \qquad \text{where } V' = \bigcap_{i=1}^l \operatorname{Ker}\tau(e_{\rho_i}).$$

Now, $\bigoplus_i V(\rho_i)$ is G-invariant by Lemma 17.2, and V' is G-invariant by (ii). By construction, we have $\bigoplus_i V(\rho_i) \subseteq V_D$ and $V' \subseteq V_{D'}$; for example, if κ is an irreducible subquotient of $\bigoplus_i V(\rho_i)$, then $\rho_i \subseteq \kappa_{|G^0}$ for some i, and hence $\kappa \in D$ by Proposition 17.3(b). This implies the assertion. □

Theorem 17.8. *Put* $\mathrm{Rep}(G)_{\mathrm{cusp}} = \prod_D \mathrm{Rep}(G)_D$*, where* D *runs through the cuspidal components of* $\mathbf{Irr}_{\mathrm{cusp}}(G)$*, and put* $\mathrm{Rep}(G)_{\mathrm{ind}} = \mathrm{Rep}(G)_{\mathbf{Irr}(G)\smallsetminus\mathbf{Irr}_{\mathrm{cusp}}(G)}$*. Then*

$$\mathrm{Rep}(G) = \mathrm{Rep}(G)_{\mathrm{cusp}} \times \mathrm{Rep}(G)_{\mathrm{ind}} = \prod_D \mathrm{Rep}(G)_D \times \mathrm{Rep}(G)_{\mathrm{ind}}.$$

In other words, $\mathbf{Irr}_{\mathrm{cusp}}(G)$ *splits the category* $\mathrm{Rep}(G)$*.*

Proof. For each congruence subgroup K_m, there are only finitely many isomorphism classes of irreducible compact G^0-representations with a non-zero K_m-fixed vector by Corollary 15.10. By Proposition 17.3, it follows that there are only finitely many cuspidal components, say, $D_1, \ldots, D_{j_m}$ which consist of all cuspidal irreducible representations with a non-zero K_m-fixed vector. Clearly, $j_{m_1} \leqslant j_{m_2}$ if $m_1 \leqslant m_2$.

Let $(V, \pi) \in \mathrm{Rep}(G)$. By Proposition 17.7 and induction, we obtain a decomposition

$$V = V_{\mathrm{cusp},m} \oplus V_{\mathrm{ind},m}, \tag{3.11}$$

where $V_{\mathrm{cusp},m} = \bigoplus_{i=1}^{j_m} V_{D_i}$, and $\mathrm{JH}(V_{\mathrm{ind},m})$ consists of those $(W, \sigma) \in \mathbf{Irr}(G)$ which are either cuspidal and satisfy $W^{K_m} = \{0\}$, or are not cuspidal.

For any $m \leqslant m'$ we have $V_{\mathrm{ind},m} = V_{\mathrm{ind},m'} \oplus (V_{\mathrm{ind},m} \cap V_{\mathrm{cusp},m'})$ by the very construction. Since clearly $V_{\mathrm{cusp},m'}^{K_m} = V_{\mathrm{cusp},m}^{K_m}$, we have $(V_{\mathrm{ind},m} \cap V_{\mathrm{cusp},m'})^{K_m} \subseteq V_{\mathrm{ind},m} \cap V_{\mathrm{cusp},m} = \{0\}$. Hence, we deduce

$$V_{\mathrm{ind},m}^{K_m} = V_{\mathrm{ind},m'}^{K_m} \qquad \text{for all } m \leqslant m'. \tag{3.12}$$

Now, consider the G-invariant subspaces

$$V_{\mathrm{cusp}} \coloneqq \bigcup_{m\geqslant 1} V_{\mathrm{cusp},m} \qquad \text{and} \qquad V_{\mathrm{ind}} \coloneqq \bigcap_{m\geqslant 1} V_{\mathrm{ind},m}$$

of V. By construction, we have $V_{\mathrm{cusp}} \in \mathrm{Rep}(G)_{\mathrm{cusp}}$ and $V_{\mathrm{ind}} \in \mathrm{Rep}(G)_{\mathrm{ind}}$. For all $m \geqslant 1$, we have $V_{\mathrm{cusp}}^{K_m} = V_{\mathrm{cusp},m}^{K_m}$ by (iii) and $V_{\mathrm{ind}}^{K_m} = V_{\mathrm{ind},m}^{K_m}$ by (3.12); so we deduce $V^{K_m} = V_{\mathrm{cusp}}^{K_m} \oplus V_{\mathrm{ind}}^{K_m}$. Since $V = \bigcup_{m\geqslant 1} V^{K_m}$, we finally obtain $V = V_{\mathrm{cusp}} \oplus V_{\mathrm{ind}}$. □

Corollary 17.9. *Fix* $m \geqslant 1$ *and suppose that* $(V, \pi) \in \mathrm{Rep}(G)$ *is generated by* V^{K_m} *as a* G*-representation. Then* $W^{K_m} \neq \{0\}$ *for all cuspidal subquotients* (W, σ) *of* (V, π)*.*

Proof. Let (W, σ) be a cuspidal subquotient of (V, π). By Theorem 17.8, we have a decomposition $V = V_{\mathrm{cusp}} \oplus V_{\mathrm{ind}}$, and (W, σ) is a subquotient of V_{cusp}. The decomposition also shows that V_{cusp} is generated by $V_{\mathrm{cusp}}^{K_m}$. In the proof of the above theorem, we showed $V_{\mathrm{cusp}}^{K_m} = V_{\mathrm{cusp},m}^{K_m} = \bigoplus_{i=1}^{j_m} V_{D_i}^{K_m}$, where $D_1, \ldots, D_{j_m}$ are the cuspidal components consisting of cuspidal irreducible representations with a non-zero K_m-fixed vector. Since V_{cusp} is generated by $V_{\mathrm{cusp}}^{K_m}$, it follows that $V_{\mathrm{cusp}} = \bigoplus_{i=1}^{j_m} V_{D_i}$. Let now (E, τ) be a cuspidal irreducible subquotient of (W, σ). Then (E, τ) is an irreducible subquotient of V_{cusp} and hence $\tau \in D_i$ for some i. This implies $E^{K_m} \neq \{0\}$. If $W' \subseteq W$ is a G-invariant subspace together with a G-equivariant surjection $W' \twoheadrightarrow E$, then $(W')^{K_m} \twoheadrightarrow E^{K_m} \neq \{0\}$ is surjective by Lemma 5.8 and hence $W^{K_m} \supseteq (W')^{K_m} \neq \{0\}$. □

Corollary 17.10. *Let $P = MN$ be a proper parabolic subgroup of G and let $(W,\sigma) \in \mathrm{Rep}(M)$. Then $\boldsymbol{i}_P^G(W,\sigma)$ does not have a cuspidal subquotient.*

Proof. Write $(V,\pi) = \boldsymbol{i}_P^G(W,\sigma)$. By Theorem 17.8 we have a decomposition $V = V_{\mathrm{cusp}} \oplus V_{\mathrm{ind}}$ such that $\mathrm{JH}(V) \cap \mathbf{Irr}_{\mathrm{cusp}}(G) = \mathrm{JH}(V_{\mathrm{cusp}})$. We thus have to show $V_{\mathrm{cusp}} = \{0\}$. By definition, we have $\boldsymbol{r}_P^G V_{\mathrm{cusp}} = \{0\}$ and hence Frobenius reciprocity (Theorem 14.3(a)) implies

$$\mathrm{Hom}_G\big(V_{\mathrm{cusp}}, \boldsymbol{i}_P^G W\big) \cong \mathrm{Hom}_M\big(\boldsymbol{r}_P^G V_{\mathrm{cusp}}, W\big) = \{0\}.$$

Hence, the inclusion $V_{\mathrm{cusp}} \hookrightarrow \boldsymbol{i}_P^G W$ is zero, which shows $V_{\mathrm{cusp}} = \{0\}$. □

§18. The Geometrical Lemma

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \dots, n_r)$ of n. Let $B := P_{(1,\dots,1)} \cap G$ and let $\mathcal{W}_G = \Sigma_n \cap G = \Sigma_{n_1} \times \cdots \times \Sigma_{n_r}$ be the Weyl group of G. We fix two parabolic subgroups $P = MN$ and $Q = LR$ of G, which for simplicity we assume to be standard, *i.e.*, P and Q contain B.

Lemma 18.1. *Put*

$$\mathcal{W}^{P,Q} := \big\{ w \in \mathcal{W}_G \,\big|\, w(L \cap B)w^{-1} \subseteq B \text{ and } w^{-1}(M \cap B)w \subseteq B \big\}.$$

(a) One has $G = \bigsqcup_{w \in \mathcal{W}^{P,Q}} PwQ$.

(b) If $w \in \mathcal{W}^{P,Q}$, then

$$M \cap wQw^{-1} = (M \cap wLw^{-1}) \cdot (M \cap wRw^{-1})$$

is a standard parabolic subgroup in M. In particular, $M \subseteq wQw^{-1}$ if and only if $M \subseteq wLw^{-1}$.

Remark. We make some remarks regarding Lemma 18.1.

(i) The decomposition in (b) holds for all $w \in \mathcal{W}_G$, but $M \cap wQw^{-1}$ is standard (if and) only if $w \in \mathcal{W}^{P,Q}\mathcal{W}_L$.

(ii) Even for $w \in \mathcal{W}^{P,Q}$, the parabolic subgroup $wQw^{-1} \subseteq G$ need not be standard.

Example. Let $G = \mathrm{GL}_3(F)$, $P = P_{(2,1)}$, $M = M_{(2,1)}$, and put $t = \begin{pmatrix} 1 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}$.

Then $\mathcal{W}^{P,P} = \{t, 1\}$, and neither $tPt^{-1} = \begin{pmatrix} * & * & * \\ 0 & * & 0 \\ * & * & * \end{pmatrix}$ nor $tMt^{-1} = \begin{pmatrix} * & 0 & * \\ 0 & * & 0 \\ * & 0 & * \end{pmatrix}$ are standard.

Sketch of the proof of Lemma 18.1. Part (a) is [Car85, Proposition 2.8.1(iii)]. For ease of notation, we assume throughout that $G = \mathrm{GL}_n(F)$. We will be using the following elementary facts:

- $\mathcal{W} \cong \Sigma_n$ is generated by the transpositions s_j, defined by $s_j(j) = j+1$, $s_j(j+1) = j$, and $s_j(i) = i$ whenever $i \notin \{j, j+1\}$. Put $S := \{s_1, \dots, s_{n-1}\}$. For each partition $\underline{n} = (n_1, \dots, n_r)$, the Weyl group $\mathcal{W}_{M_{\underline{n}}}$ is generated by $S_{M_{\underline{n}}} = S \cap M_{\underline{n}}$.
- For each $w \in \mathcal{W}$, denote $\mathrm{inv}(w) := \{(i,j) \,|\, i < j \text{ and } w(i) > w(j)\}$ the set of inversions. The number $\ell(w) := \#\,\mathrm{inv}(w)$ is called the *length* of w. If $(j, j+1)$ is an inversion of w, then the map

$$\begin{aligned} \mathrm{inv}(w) \smallsetminus \{(j,j+1)\} &\xrightarrow{\cong} \mathrm{inv}(ws_i), \\ (i_1, i_2) &\longmapsto (s_j(i_1), s_j(i_2)) \end{aligned}$$

is bijective; in particular, $\#\operatorname{inv}(w) = \#\operatorname{inv}(ws_j) + 1$. By induction, we deduce that $\ell(w)$ is the smallest integer r such that there exist $1 \leqslant j_1, \dots, j_r \leqslant n-1$ with $w = s_{j_1} \cdots s_{j_r}$.

– For each $1 \leqslant i \neq j \leqslant n$, put $U_{(i,j)} := \{e_{ij}(\lambda) \mid \lambda \in F\}$, where $e_{ij}(\lambda)$ is an elementary matrix (see the proof of the Bruhat decomposition 12.9). Then $\operatorname{inv}(w)$ is the set of those pairs (i,j) with $U_{(i,j)} \subseteq B$ and $wU_{(i,j)}w^{-1} \subseteq \overline{B}$. Moreover, U is generated as a group by $U_{(1,2)}, \dots, U_{(n-1,n)}$.

Note that, since $B \subseteq P$ and $B \subseteq Q$, each double coset PgQ is a union of cosets of the form BwB, where $w \in \mathcal{W}_G$, by the Bruhat decomposition 12.9. Hence, there exists a subset $X \subseteq \mathcal{W}_G$ such that $G = \bigsqcup_{w \in X} PwQ$.

We first argue that X is a representing system for $\mathcal{W}_M \backslash \mathcal{W}_G / \mathcal{W}_L$. Observe that $\mathcal{W}_G \cap P = \mathcal{W}_M$. The Bruhat decomposition 12.9 implies $P = B\mathcal{W}_M B$. Similarly, we have $Q \cap \mathcal{W}_G = \mathcal{W}_L$ and $Q = B\mathcal{W}_L B$. For each $w \in \mathcal{W}_G$, we thus need to show

$$B\mathcal{W}_M BwB\mathcal{W}_L B \subseteq B\mathcal{W}_M w \mathcal{W}_L B.$$

This follows inductively from the following fact:

Fact. For each $1 \leqslant j \leqslant n-1$ and $w \in \mathcal{W}$, one has $s_j B w \subseteq Bs_j wB \sqcup BwB$ and symmetrically, $wBs_j \subseteq Bws_j B \sqcup BwB$.

Proof of the fact: We only prove the first inclusion. The second follows from the first by passing to inverses. Put $B' := wBw^{-1}$. Then it suffices to show $s_j B \subseteq Bs_j B' \sqcup BB'$. Let $e_1, \dots, e_n$ be the standard basis of F^n and denote $G_j \subseteq G = \mathrm{GL}_n(F)$ the subgroup of elements which fix e_i, whenever $i \notin \{j, j+1\}$, and which stabilize $Fe_j + Fe_j$; then $G_j \cong \mathrm{GL}_2(F)$. One easily checks $s_j \in G_j$ and $G_j B = P_{(1,\dots,1,2,1,\dots,1)} = BG_j$, where the 2 is in the j-th spot. Hence, $s_j B \subseteq BG_j$, and it remains to prove

$$G_j \subseteq (B \cap G_j)s_j(B' \cap G_j) \sqcup (B \cap G_j)(B' \cap G_j). \tag{3.13}$$

Note that $B_2 := B \cap G_j$ corresponds to the group of upper triangular matrices in $\mathrm{GL}_2(F)$. If $w^{-1}(j) < w^{-1}(j+1)$, then $U_{(j,j+1)} \subseteq wBw^{-1}$ and hence $B' \cap G_j = B_2$. Otherwise, one has $U_{(j+1,j)} \subseteq wBw^{-1}$ and hence $B' \cap G_j =: \overline{B}_2$ corresponds to the group of lower triangular matrices in $\mathrm{GL}_2(F)$. By the Bruhat decomposition 12.9, we have $G_j = B_2 \sqcup B_2 s_j B_2$. Multiplying from the right with s_j^{-1}, we deduce $G_j = B_2 s_j \overline{B}_2 \sqcup B_2 \overline{B}_2$, which proves (3.13). □

We now know that X is a representing system for $\mathcal{W}_M \backslash \mathcal{W}_G / \mathcal{W}_L$. We choose X such that each $w \in X$ has minimal length in $\mathcal{W}_M w \mathcal{W}_L$. We claim $X = \mathcal{W}^{P,Q}$. Let $w \in X$. For each j with $s_j \in S_L$, we then have $w(j) < w(j+1)$, because otherwise ws_j would be a representative of $\mathcal{W}_M w \mathcal{W}_L$ of smaller length than w. Hence, $wU_{(j,j+1)}w^{-1} \subseteq B$ and then, since $L \cap U$ is generated by the $U_{(j,j+1)}$ with $s_j \in S_L$, also $w(L \cap B)w^{-1} \subseteq B$. A similar argument shows $w^{-1}(M \cap B)w \subseteq B$, whence $X \subseteq \mathcal{W}^{P,Q}$.

In order to prove $\mathcal{W}^{P,Q} \subseteq X$, it suffices to show that $\mathcal{W}_M w \mathcal{W}_L \cap \mathcal{W}^{P,Q}$ contains at most one element, for all $w \in \mathcal{W}_G$. This is the content of the following claim.

Claim 1. Let $v, w \in \mathcal{W}^{P,Q}$ and $x \in \mathcal{W}_M$, $y \in \mathcal{W}_L$ with $xv = wy$.

(i) One has $x = 1$ if and only if $y = 1$.

(ii) If $x \neq 1$, there exists $s_j \in v^{-1}S_M v \cap S_L$ such that $\ell(xs_{v(j)}) < \ell(x)$ and $\ell(ys_j) < \ell(y)$.

(iii) One has $v = w$.

(iv) $\mathcal{W}_M \cap w\mathcal{W}_L w^{-1}$ is generated as a group by $S_M \cap wS_Lw^{-1}$.

Proof of the claim. Note that $\mathcal{W}^{P,Q}$ is the set of all $w \in \mathcal{W}_G$ with $w(j) < w(j+1)$ for all j with $s_j \in S_L$, and $w^{-1}(i) < w^{-1}(i+1)$ for all i with $s_i \in S_M$; this follows from the fact that $L \cap B$ (resp. $M \cap B$) is generated by T and the $U_{(j,j+1)}$ with $s_j \in S_L$ (resp. by T and the $U_{(i,i+1)}$ with $s_i \in S_M$).

We first prove (i). Let $x = 1$ and assume for a contradiction that $y \neq 1$. Then there exists $s_j \in S_L$ such that $y(j) > y(j+1)$. Note that $U_{(y(j),y(j+1))} = yU_{(j,j+1)}y^{-1} \subseteq L \cap B$ and hence $w \in \mathcal{W}^{P,Q}$ implies $v(j) = wy(j) > wy(j+1) = v(j+1)$, which contradicts $v \in \mathcal{W}^{P,Q}$. This shows that $x = 1$ implies $y = 1$. A similar argument shows the reverse implication.

We now prove (ii). Assume $x \neq 1$. By (i) we have $y \neq 1$. Hence, there exists $s_j \in S_L$ such that $y(j) > y(j+1)$. Then clearly $\ell(ys_j) < \ell(y)$. Since $v, w \in \mathcal{W}^{P,Q}$, we have $v(j) < v(j+1)$ and $xv(j) = wy(j) > wy(j+1) = xv(j+1)$. Therefore, $(v(j), v(j+1))$ is an inversion of x. But since $x \in \mathcal{W}_M$, we deduce $U_{(v(j),v(j+1))} \subseteq M \cap B$. We claim $v(j+1) = v(j) + 1$, which then implies $s_{v(j)} \in S_M$ and $\ell(xs_{v(j)}) < \ell(x)$: Indeed, if we had $v(j) < i < v(j+1)$ for some i, then $U_{(v(j),i)}, U_{(i,v(j+1))} \subseteq M \cap B$, and then $v \in \mathcal{W}^{P,Q}$ implies $j = v^{-1}(v(j)) < v^{-1}(i) < v^{-1}(v(j+1)) = j+1$, a contradiction. This finishes the proof (ii).

We prove (iii) by induction on $\ell(x)$. If $x = 1$, then $y = 1$ by (i), and hence $v = w$. Let now $x \neq 1$. By (ii) there exists $s_j \in v^{-1}S_Mv \cap S_L$ such that $\ell(x') < \ell(x)$ and $\ell(y') < \ell(y)$, where $x' \coloneqq xs_{v(j)}$ and $y' \coloneqq ys_j$. Since $s_{v(j)} = vs_jv^{-1}$, we compute

$$x'v = xs_{v(j)}v = xvs_j = wys_j = wy'.$$

By the induction hypothesis, we conclude $v = w$.

The same argument proves (iv). Denote $\mathcal{W}'$ the group generated by $S_M \cap wS_Lw^{-1}$. Let $x \in \mathcal{W}_M$ and $y \in \mathcal{W}_L$ such that $x = wyw^{-1} \in \mathcal{W}_M \cap w\mathcal{W}_Lw^{-1}$. We show $x \in \mathcal{W}'$ by induction on $\ell(x)$. If $\ell(x) = 0$, there is nothing to show. If $x \neq 1$, then also $y \neq 1$ by (ii). Therefore, we find $s_j \in w^{-1}S_Mw \cap S_L$ such that for $x' \coloneqq xs_{w(j)}$ and $y' \coloneqq ys_j$, we have $\ell(x') < \ell(x)$ and $\ell(y') < \ell(y)$. As before, we deduce $x' = wy'w^{-1} \in \mathcal{W}_M \cap w\mathcal{W}_Lw^{-1}$. By the induction hypothesis, we have $x' \in \mathcal{W}'$ and hence also $x = x's_{w(j)} \in \mathcal{W}'$. □

We now prove (b). The set $S_M \cap wS_Lw^{-1}$ determines a partition $\underline{n}'$. We show $M \cap P_{\underline{n}'} = M \cap wQw^{-1}$. Since $w \in \mathcal{W}^{P,Q}$, we have $w^{-1}(M \cap B)w \subseteq B \subseteq Q$ and hence $M \cap B \subseteq M \cap wQw^{-1}$. Together with $M_{\underline{n}'} \subseteq M \cap wLw^{-1}$, we deduce $M \cap P_{\underline{n}'} \subseteq M \cap wQw^{-1}$. Moreover, we have $\mathcal{W}_G \cap M \cap wQw^{-1} = \mathcal{W}_M \cap w\mathcal{W}_Lw^{-1} = \mathcal{W}_{M_{\underline{n}}}$ by Claim 1(iv). The Bruhat decomposition 12.9 shows the reverse inclusion:

$$M \cap wQw^{-1} = \bigsqcup_{v \in \mathcal{W}_{M_{\underline{n}'}}} (M \cap B)v(M \cap B) \subseteq M \cap P_{\underline{n}'}.$$

Let now $i < j$ such that $U_{(i,j)} \subseteq M \cap wLw^{-1}$. Then $M \cap wLw^{-1}$ also contains $U_{(j,i)}$ and the computation

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

shows that the transposition which interchanges i and j belongs to $\mathcal{W}_M \cap w\mathcal{W}_Lw^{-1} = \mathcal{W}_{M_{\underline{n}'}}$. But this implies $U_{(i,j)} \subseteq M_{\underline{n}'}$. The contrapositive shows $M \cap U_{\underline{n}'} \subseteq M \cap wRw^{-1}$. Now,

$$M \cap wQw^{-1} = M \cap P_{\underline{n}'} = M_{\underline{n}'} \cdot (M \cap U_{\underline{n}'}) \subseteq (M \cap wLw^{-1}) \cdot (M \cap wRw^{-1}) \subseteq M \cap wQw^{-1}.$$

Hence, we have equality throughout, and $M_{\underline{n}'} = M \cap wLw^{-1}$ and $M \cap U_{\underline{n}'} = M \cap wRw^{-1}$. □

Compare the following theorem with the Mackey decomposition 9.5.

Theorem 18.2 (Geometrical Lemma). *Let $(W, \sigma) \in \operatorname{Rep}(M)$. There exist an ordering $\mathcal{W}^{P,Q} = \{w_1, \dots, w_l\}$ and a filtration*

$$\{0\} = F_0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_l = \boldsymbol{r}_Q^G \, \boldsymbol{i}_P^G(W, \sigma)$$

by L-invariant subspaces together with L-equivariant isomorphisms

$$F_i/F_{i-1} \cong \boldsymbol{i}^L_{w_i^{-1} P w_i \cap L} \, w_{i*}^{-1} \, \boldsymbol{r}^M_{M \cap w_i Q w_i^{-1}}(W, \sigma), \quad \textit{for all } 1 \leqslant i \leqslant l.$$

Sketch of the proof. A detailed proof can be found in [Ren10, VI.5.1]. We briefly explain how to construct the filtration. Let $P \times Q$ act on G via $(x, y) \cdot g := xgy^{-1}$. For every $P \times Q$-invariant subset $Y \subseteq G$ and each $(E, \tau) \in \operatorname{Rep}(P)$ we put

$$\operatorname{ind}_P^Y(E, \tau) := \left\{ f \colon Y \to E \;\middle|\; \begin{array}{l} f(gy) = \tau(g) f(y) \text{ for all } g \in P,\, y \in Y, \\ f \text{ is locally constant, and} \\ \text{the image of } \operatorname{Supp} f \text{ in } P \backslash Y \text{ is compact} \end{array} \right\} \in \operatorname{Rep}(Q).$$

Choose an ordering $\mathcal{W}^{P,Q} = \{w_1, \dots, w_l\}$ such that the subsets

$$Y_i := \bigsqcup_{j=1}^{i} P w_j Q \subseteq G$$

are open, for all $1 \leqslant i \leqslant l$. For $(W, \sigma) \in \operatorname{Rep}(M)$, the filtration in the assertion is then given by

$$F_i := J_R\big(\delta_Q^{-1/2} \otimes \operatorname{ind}_P^{Y_i}(\delta_P^{1/2} \otimes \operatorname{Inf}_P^M \sigma)\big) \subseteq \boldsymbol{i}_P^G \, \sigma \qquad \text{for } 1 \leqslant i \leqslant l.$$

Let us now sketch the argument for why we have $F_i/F_{i-1} \cong \boldsymbol{i}^L_{w_i^{-1} P w_i \cap L} \, w_{i*}^{-1} \, \boldsymbol{r}^M_{M \cap w_i Q w_i^{-1}}(W, \sigma)$. To lighten the notation, we write $Y := Y_{i-1}$, $Y' := Y_i$ and $w := w_i$.

Claim 1. (a) We have a short exact sequence

$$0 \longrightarrow \operatorname{ind}_P^Y W \longrightarrow \operatorname{ind}_P^{Y'} W \longrightarrow \operatorname{ind}_P^{PwQ} W \longrightarrow 0,$$

where the first and second maps are given by extension by zero and restriction of functions, respectively.

(b) The map

$$\begin{aligned} \operatorname{ind}_P^{PwQ} \sigma &\xrightarrow{\cong} \operatorname{ind}^Q_{w^{-1} P w \cap Q} \, w_*^{-1} \sigma_{|P \cap wQw^{-1}}, \\ f &\longmapsto [q \mapsto f(wq)] \end{aligned}$$

is a Q-equivariant isomorphism.

(c) Write $P' = w^{-1}Pw$ and denote the projection map $W \to J_{P'\cap R}(W)$ by $v \mapsto \overline{v}$. Let $\delta := (\delta_R)_{|P'\cap Q} \otimes \delta^{-1}_{P'\cap R}$, which is a smooth character of $P' \cap Q$.[4] The map

$$\begin{aligned} J_R \operatorname{ind}^Q_{P'\cap Q} \sigma &\xrightarrow{\cong} \operatorname{ind}^L_{P'\cap L}\big(\delta \otimes J_{P'\cap R}(\sigma)\big), \\ f &\longmapsto \Big[g \mapsto \int_{(P'\cap R)\backslash R} \overline{f(xg)}\, \mathrm{d}\nu(x)\Big] \end{aligned}$$

is an L-equivariant isomorphism. Here, ν denotes a semi-invariant Haar measure on the space $(P' \cap R)\backslash R$.

Proof of the claim. Part (b) is proved in the same way as (2.12) in the proof of the Mackey decomposition 9.5. For the support conditions, one has to check that the inclusion $Q \hookrightarrow w^{-1}PwQ$ and multiplication $w^{-1}PwQ \xrightarrow{w\cdot} PwQ$ induce homeomorphisms $w^{-1}Pw \cap Q\backslash Q \xrightarrow{\cong} w^{-1}Pw\backslash w^{-1}PwQ \xrightarrow{\cong} P\backslash PwQ$ on the right coset spaces (which come equipped with the quotient topologies).

If $Z \subseteq G$ is any $P \times Q$-invariant subset, the multiplication map

$$\begin{aligned} C_c^\infty(P\backslash Z) \otimes_{\mathbb{C}} W &\xrightarrow{\cong} \operatorname{ind}^Z_P W, \\ f \otimes v &\longmapsto [z \mapsto f(z)v] \end{aligned}$$

is a $\mathbb{C}$-linear isomorphism, where $C_c^\infty(P\backslash Z)$ is the space of all locally constant functions $P\backslash Z \to \mathbb{C}$ (which we also view as functions $Z \to \mathbb{C}$ which are invariant under left translation by P) with compact support; see the proof of (2.6). Hence, the sequence in (a) arises from the sequence

$$0 \longrightarrow C_c^\infty(P\backslash Y) \longrightarrow C_c^\infty(P\backslash Y') \longrightarrow C_c^\infty(P\backslash PwQ) \longrightarrow 0 \tag{3.14}$$

by applying the exact functor $\_ \otimes_{\mathbb{C}} W$. Hence, it suffices to show that (3.14) is exact. Exactness on the left and in the middle are clear, so it remains to prove that the restriction of functions yields a surjective map $C_c^\infty(P\backslash Y') \to C_c^\infty(P\backslash PwQ)$. So let $f\colon P\backslash PwQ \to \mathbb{C}$ be locally constant with compact support. We find a compact open subgroup $H \subseteq G$ and $\{z_j\}_{j\in J} \subseteq G$ such that $G = \bigsqcup_{j\in J} Pz_jH$ and f is constant with value, say, c_j on the subsets $Pz_jH \cap PwQ$; we put $c_j := 0$ if $Pz_jH \cap PwQ = \varnothing$. Since f has compact support, only finitely many of the c_j are non-zero. Let now $f'\colon Y' \to \mathbb{C}$ be the function which is constant on Pz_jH with value c_j, for all $j \in J$. Then f' lies in $C_c^\infty(Y')$ and satisfies $f'_{|PwQ} = f$.

For part (c), we refer to [Cas95, Proposition 6.2.1]. We fix left invariant Haar measures μ_R and $\mu_{P'\cap R}$ on R and $P' \cap R$, respectively. Let $\nu\colon C_c^\infty((P' \cap R)\backslash R, \theta = 1) \to \mathbb{C}$ be the associated semi-invariant Haar measure; note that the modulus characters of R and $P' \cap R$ are trivial, since both groups are unions of its compact open subgroups (Example 11.1). For each $\mathbb{C}$-vector space E on which R acts trivially, we obtain an E-valued Haar measure

$$C_c^\infty\big((P' \cap R)\backslash R, E\big) \cong C_c^\infty\big((P' \cap R)\backslash R\big) \otimes_{\mathbb{C}} E \xrightarrow{\nu\otimes \mathrm{id}_E} E,$$

which we again denote ν. We observe the following properties of ν:

[4] In fact, we need to extend the notion of *modulus character* a bit: for each $g \in Q$ denote $\mathrm{conj}_g\colon R \to R$, $x \mapsto gxg^{-1}$ the conjugation by g. Then $\mu'_R(f) := \mu_R(f \circ \mathrm{conj}_g^{-1})$ defines another left invariant Haar measure on R, and hence $\mu'_R = \delta_R(g)\mu_R$ for some $\delta_R(g) \in \mathbb{R}_{>0}$. It is easy to see that $\delta_R\colon Q \to \mathbb{R}^\times_{>0}$ is a smooth character. Similarly, we extend $\delta_{P'\cap R}$ to a smooth character $P' \cap Q \to \mathbb{R}^\times_{>0}$.

– Let $g \in P' \cap L$ and denote $\mathrm{conj}_g \colon R \to R$, $x \mapsto gxg^{-1}$. Then $\mathrm{conj}_g(P' \cap R) = P' \cap R$, and $\nu'(f) \coloneqq \nu(f \circ \mathrm{conj}_g^{-1})$ defines another semi-invariant Haar measure. Hence, there exists a scalar $\delta_{(P'\cap R)\backslash R}(g) \in \mathbb{C}^\times$ such that $\nu' = \delta_{(P'\cap R)\backslash R}(g) \cdot \nu$.

  Exercise: show that $\delta_{(P'\cap R)\backslash R}(g) = \delta(g) = \delta_R(g)\delta_{P'\cap R}(g)^{-1}$.

– If $\alpha \colon E \to E$ is a $\mathbb{C}$-linear automorphism, then $\nu(\alpha \circ f) = \alpha(\nu(f))$.

We check that the map

$$\begin{aligned} \mathrm{ind}_{P'\cap Q}^{Q}(W,\sigma) &\longrightarrow \mathrm{Inf}_Q^L \, \mathrm{ind}_{P'\cap L}^{L}\big(\delta \otimes J_{P'\cap R}(W,\sigma)\big), \\ f &\longmapsto \overline{f} \coloneqq \Big[g \mapsto \int_{(P'\cap R)\backslash R} \overline{f(xg)} \, \mathrm{d}\nu(x) \Big] \end{aligned}$$

is well-defined. Let $f \in \mathrm{ind}_{P'\cap Q}^{Q} W$. For each $g \in Q$, the map $R \to J_{P'\cap R}(W)$, $x \mapsto \overline{f(xg)}$ is locally constant with compact support modulo $P' \cap R$, and lies in $C_c^\infty((P'\cap R)\backslash R, J_{P'\cap R}(W))$. Hence, the integral $\int_{(P'\cap R)\backslash R} \overline{f(xg)} \, \mathrm{d}\nu(x)$ is well-defined for each g. Since ν is right invariant with respect to R, the integral depends only on the image of g in $Q/R \cong L$; in particular, R acts trivially on $\overline{f}$ under right translation. For each $y \in P' \cap L$ and $g \in L$ we compute

$$\begin{aligned} \overline{f}(yg) &= \int_{(P'\cap R)\backslash R} \overline{f(xyg)} \, \mathrm{d}\nu(x) = \int_{(P'\cap R)\backslash R} \overline{\sigma(y) f(y^{-1}xy \cdot g)} \, \mathrm{d}\nu(x) \\ &= \delta(y) J_{P'\cap R}(\sigma)(y) \int_{(P'\cap R)\backslash R} \overline{f(xg)} \, \mathrm{d}\nu(x) = (\delta \otimes J_{P'\cap R}(\sigma))(y) \overline{f}(g). \end{aligned}$$

Note that $\overline{f}$ is locally constant and has compact support modulo $P' \cap L$. We deduce that $\overline{f}$ lies in $\mathrm{Inf}_Q^L \, \mathrm{ind}_{P'\cap L}^{L}\big(\delta \otimes J_{P'\cap R}(W,\sigma)\big)$. It is clear that the map $f \mapsto \overline{f}$ is Q-linear and hence induces an L-equivariant map

$$\Phi \colon J_R \, \mathrm{ind}_{P'\cap Q}^{Q}(W,\sigma) \longrightarrow \mathrm{ind}_{P'\cap L}^{L}\big(\delta \otimes J_{P'\cap R}(W,\sigma)\big).$$

We show that Φ is surjective: It suffices to exhibit a generating set of $\mathrm{ind}_{P'\cap L}^{L}(\delta \otimes J_{P'\cap R}(\sigma))$ which lies in the image of Φ. For any triple (w, g, K), where $K \subseteq L$ is a compact open subgroup, $g \in L$, and $w \in J_{P'\cap R}(W)^{(P'\cap L)\cap gKg^{-1}}$, we denote $f_{w,g,K} \colon L \to J_{P'\cap R}(W)$ the function with support $(P' \cap L)gK$ given by

$$f_{w,g,K}(xgk) \coloneqq \delta(x) J_{P'\cap R}(\sigma)(x) w,$$

for all $x \in P' \cap L$ and $k \in K$. (Check that $xgk = x'gk'$ implies $f_{w,g,K}(xgk) = f_{w,g,K}(x'gk')$, which requires that w is fixed by $(P' \cap L) \cap gKg^{-1}$.) It is clear that the $f_{w,g,K}$, where K runs through a fundamental system of compact open subgroups, span $\mathrm{ind}_{P'\cap L}^{L}(\delta \otimes J_{P'\cap R}(W))$ as a $\mathbb{C}$-vector space. Let $K_0 \subseteq Q$ be a compact open subgroup with image K in L. It remains to show that $f_{w,g,K}$ lies in the image of Φ. The image of $P' \cap Q \cap gK_0g^{-1}$ in L then coincides with $P' \cap L \cap gKg^{-1}$. Since the quotient map $W \to \mathrm{Inf}_{P'\cap Q}^{P'\cap L} J_{P'\cap R} W$ is surjective and taking $P' \cap Q \cap gK_0g^{-1}$-invariants is exact by Lemma 5.8, we may pick a lift $w_0 \in W^{P'\cap Q\cap gK_0g^{-1}}$ of w. Consider now the function $f \colon Q \to W$ with support $(P' \cap Q)gK_0$ given by $f(xgk) = \sigma(x)w_0$ for all $x \in P' \cap Q$ and $k \in K_0$. We claim $\Phi(f) = c \cdot f_{w,g,K}$ for some $c > 0$. Let $y \in R$ with $yg \in (P' \cap Q)gK_0$, and pick $x \in P' \cap Q$

and $k \in K_0$ such that $yg = xgk$. Denoting $\mathrm{pr}_L\colon Q \to Q/R \cong L$ the projection, we compute $g = \mathrm{pr}_L(yg) = \mathrm{pr}_L(xgk) = \mathrm{pr}_L(x) g \,\mathrm{pr}_L(k)$, and hence $\mathrm{pr}_L(x) \in (P' \cap L) \cap gKg^{-1}$. We deduce

$$\overline{f(yg)} = \overline{f(xgk)} = \overline{\sigma(x)w_0} = J_{P'\cap R}(\sigma)(x)w = J_{P'\cap R}(\sigma)(\mathrm{pr}_L(x))w = w.$$

Therefore, $\Phi(f)(g) = \int_{(P'\cap R)\backslash R} \overline{f(xg)}\,\mathrm{d}\nu(x) = c \cdot w$, where $c = \nu_{(P'\cap R)\backslash R}(\mathbf{1}_{R\cap(P'\cap Q)gK_0g^{-1}}) > 0$. Moreover, it is clear from the definition that $\Phi(f)$ is fixed by K and that the support of $\Phi(f)$ is $(P' \cap L)gK$. It follows that $\Phi(f) = c \cdot f_{w,g,K}$, which shows that Φ is surjective.

It remains to prove that Φ is injective. We need the following claim:

Claim 2. Recall the left Haar measure $\mu_R\colon C_c^\infty(R) \to \mathbb{C}$ on R.

(a) Let $R_0 \subseteq R$ and $K \subseteq Q$ be compact open subgroups. Then K normalizes a compact open subgroup $R_1 \subseteq R$ containing R_0.

(b) Let $f \in \mathrm{ind}_{P'\cap Q}^Q W$. For each $g \in Q$ we define $f_g \in C_c^\infty(R, W)$ as $f_g(x) \coloneqq f(gx)$. Then $f \in (\mathrm{ind}_{P'\cap Q}^Q W)(R)$ if and only if for every $g \in Q$ there exists a compact open subgroup $R_g \subseteq R$ such that $\rho(e_{R_g})f_g = 0$.

Proof of the claim: We first show (a). The subset $X \coloneqq \bigcup_{k\in K} kR_0k^{-1} \subseteq R$ is compact as the image of the map $K \times R_0 \to R$, $(k, x) \mapsto kxk^{-1}$. Let $R_0' \subseteq R$ be a compact open subgroup containing X (which is possible by Remark 12.16). Then $R_1 \coloneqq \bigcap_{k\in K} kR_0'k^{-1}$ is a compact subgroup normalized by K. By construction, R_1 contains R_0 and hence is open.

We prove (b). Suppose $f \in (\mathrm{ind}_{P'\cap Q}^Q W)(R)$. Since R is the union of its compact open subgroups, we find a compact open subgroup $R_0 \subseteq R$ with $f \in (\mathrm{ind}_{P'\cap Q}^Q W)(R_0)$. By Lemma 7.8, we find $\rho(e_{R_0})f_g = \big(\rho(e_{R_0})f\big)(g) = 0$.

We now prove the converse direction. Let $K \subseteq Q$ be a compact open subgroup fixing f. As f has compact support, we find $g_1, \dots, g_r \in Q$ with $\mathrm{Supp}(f) = \bigsqcup_{i=1}^r (P' \cap Q)g_iK$. By (a), applied to a compact open subgroup R_0 containing $R_{g_1}, \dots, R_{g_r}$, we find a compact open subgroup $R_1 \subseteq R$ which is normalized by K and contains R_{g_i} for all i. Then $\rho(e_{R_1})f_{g_i} = \rho(e_{R_1} * e_{R_{g_i}})f_{g_i} = \rho(e_{R_1})\rho(e_{R_{g_i}})f_{g_i} = 0$ for all i, where we have used Proposition 7.4(a) for the first equality. Let now $g \in Q$ be arbitrary. If $f(gz) = 0$ for all $z \in R_1$, then $\rho(e_{R_1})f_g = 0$. Otherwise, we find $z \in R_1$, $x \in P' \cap Q$, $k \in K$, and $1 \leqslant i \leqslant r$ such that $gz = xg_ik$. Since K normalizes R_1 and fixes f, and because μ_R is left invariant, we compute

$$\begin{aligned}
\mathrm{vol}(R_1;\mu_R) \cdot \rho(e_{R_1})f_g &= \int_R f(xg_ikz^{-1}y)\mathbf{1}_{R_1}(y)\,\mathrm{d}\mu_R(y) = \int_R f(xg_iky)\mathbf{1}_{R_1}(zy)\,\mathrm{d}\mu_R(y) \\
&= \sigma(x)\int_R f(g_ikyk^{-1})\mathbf{1}_{R_1}(y)\,\mathrm{d}\mu_R(y) \\
&= \delta_R(k)^{-1}\sigma(x)\int_R f(g_iy)\mathbf{1}_{R_1}(k^{-1}yk)\,\mathrm{d}\mu_R(y) \\
&= \delta_R(k)^{-1}\sigma(x)\rho(e_{R_1})f_{g_i} = 0.
\end{aligned}$$

This shows $\rho(e_{R_1})f = 0$ and hence $f \in (\mathrm{ind}_{P'\cap Q}^Q W)(R_1) \subseteq (\mathrm{ind}_{P'\cap Q}^Q W)(R)$ by Lemma 7.8. □

Let now $f \in \operatorname{ind}_{P'\cap Q}^{Q} W$ with $\overline{f} = 0$, and let $g \in Q$ be fixed but arbitrary. The function $(\rho(g)f)_{|R}$ has compact support, and hence we find a compact open subgroup $R_0 \subseteq R$ such that $\operatorname{Supp}(\rho(g)f)_{|R} \subseteq (P' \cap R)R_0$. By the definition of ν we compute

$$\begin{aligned}\int_R \overline{f(xg)}\mathbf{1}_{R_0}(x)\,\mathrm{d}\mu_R(x) &= \int_{(P'\cap R)\backslash R}\int_{P'\cap R} \overline{f(xyg)}\mathbf{1}_{R_0}(xy)\,\mathrm{d}\mu_{P'\cap R}(x)\mathrm{d}\nu(y)\\ &= \int_{(P'\cap R)\backslash R}\int_{P'\cap R} \overline{f(yg)}\mathbf{1}_{R_0}(xy)\,\mathrm{d}\mu_{P'\cap R}(x)\mathrm{d}\nu(y)\\ &= \int_{(P'\cap R)\backslash R} \operatorname{vol}(P'\cap R_0 y^{-1};\mu_{P'\cap R})\cdot\overline{f(yg)}\,\mathrm{d}\nu(y)\\ &= \operatorname{vol}(P'\cap R_0;\mu_{P'\cap R})\overline{f}(g) = 0,\end{aligned}$$

where for the fourth equality we have used $\operatorname{Supp}(\rho(g)f)_{|R} \subseteq (P'\cap R)R_0$ and that, for $y_1 \in P'\cap R$ and $y_2 \in R_0$, we have $\operatorname{vol}(P'\cap R_0(y_1y_2)^{-1}) = \operatorname{vol}(P'\cap R_0 y_1^{-1}) = \delta_{P'\cap R}(y_1)^{-1}\operatorname{vol}(P'\cap R_0) = \operatorname{vol}(P'\cap R_0)$, since $P'\cap R$ is unimodular. We deduce

$$\begin{aligned}\operatorname{vol}(g^{-1}R_0 g\ \mu_R)\cdot\rho(e_{g^{-1}R_0g})f_g &= \int_R f(gx)\mathbf{1}_{g^{-1}R_0g}(x)\,\mathrm{d}\mu_R(x)\\ &= \int_R f(gxg^{-1}\cdot g)\mathbf{1}_{R_0}(gxg^{-1})\,\mathrm{d}\mu_R(x)\\ &= \delta_R(g^{-1})\int_R f(xg)\mathbf{1}_{R_0}(x)\,\mathrm{d}\mu_R(x) \in W(R).\end{aligned}$$

As R is the union of its compact open subgroups, we find $R_g \subseteq R$ containing $g^{-1}R_0g$, and such that $\rho(e_{g^{-1}R_0g})f_g \in W(R_1)$. We now have $\rho(e_{R_g})f_g = \rho(e_{R_g})\rho(e_{g^{-1}R_0g})f_g = 0$. Hence, the criterion in Claim 2(b) is satisfied and shows $f \in (\operatorname{ind}_{P'\cap Q}^{Q} W)(R)$. This shows that Φ is injective and finishes the proof of (c). □

Using Claim 1, we now compute

$$\begin{aligned}F_i/F_{i-1} &\cong J_R\big(\delta_Q^{-1/2}\otimes\operatorname{ind}_P^{Pw_iQ}(\delta_P^{1/2}\otimes\operatorname{Inf}_P^M\sigma)\big) && \text{(Claim (a))}\\ &\cong J_R\big(\delta_Q^{-1/2}\otimes\operatorname{ind}_{w_i^{-1}Pw_i\cap Q}^{Q} w_{i*}^{-1}\delta_P^{1/2}\otimes w_{i*}^{-1}\operatorname{Inf}_P^M\sigma\big) && \text{(Claim (b))}\\ &\cong \operatorname{ind}_{w_i^{-1}Pw_i\cap L}^{L}\big(\delta_Q^{-1/2}\otimes w_{i*}^{-1}\delta_P^{1/2}\otimes\delta\otimes w_{i*}^{-1}J_{P\cap w_iRw_i^{-1}}(\operatorname{Inf}_P^M\sigma)\big) && \text{(Claim (c))}\\ &\cong \operatorname{ind}_{w_i^{-1}Pw_i\cap L}^{L}\big(\delta_Q^{-1/2}\otimes w_{i*}^{-1}\delta_P^{1/2}\otimes\delta\otimes\operatorname{Inf}_{w_i^{-1}Pw_i\cap L}^{w_i^{-1}Mw_i\cap L} w_{i*}^{-1}J_{M\cap w_iRw_i^{-1}}(\sigma)\big),\end{aligned}$$

where $\delta = \delta_{(w_i^{-1}Pw_i\cap L)R}\otimes\delta_{w_i^{-1}Pw_i\cap Q}^{-1}$, viewed as a character of $w_i^{-1}Pw_i\cap L$. One finally needs to show that

$$\delta_Q^{-1/2}\otimes w_{i*}^{-1}\delta_P^{1/2}\otimes\delta = \delta_{w_i^{-1}Pw_i\cap L}^{1/2}\otimes\delta_{w_i^{-1}Mw_i\cap Q}^{-1/2},$$

from which we obtain $F_i/F_{i-1} \cong \boldsymbol{i}_{w_i^{-1}Pw_i\cap L}^{L}\, w_{i*}^{-1}\,\boldsymbol{r}_{M\cap w_iQw_i^{-1}}^{M}(\sigma)$. □

§19. Finiteness Theorems

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \ldots, n_r)$ of n. Recall from Theorem 14.3(d)/(c) that the functor $\boldsymbol{i}_P^G$ preserves admissibility and $\boldsymbol{r}_P^G$ preserves finite generation. In this section, we will show that $\boldsymbol{i}_P^G$ preserves finite length, and $\boldsymbol{r}_P^G$ preserves admissibility and finite length.

Theorem 19.1. *Let $P = MN$ be a parabolic subgroup of G, and let $(W, \sigma) \in \operatorname{Rep}(M)$ have finite length. Then $\boldsymbol{i}_P^G(W, \sigma)$ has finite length.*

Proof. Let $g \in G$ such that gPg^{-1} is standard. The map

$$\begin{aligned} \boldsymbol{i}_P^G(W, \sigma) &\longrightarrow \boldsymbol{i}_{gPg^{-1}}^G(W, g_*\sigma), \\ f &\longmapsto [\gamma \mapsto f(g^{-1}\gamma)] \end{aligned}$$

is clearly an isomorphism. Hence, we may assume from the start that P is standard.

Since $\boldsymbol{i}_P^G$ is exact by Theorem 14.3(b), we may assume that (W, σ) is irreducible.

By Lemma 15.2 there exists a proper parabolic subgroup $Q = LR$ of M and a cuspidal representation $(E, \tau) \in \operatorname{Rep}(L)$ such that $(W, \sigma) \subseteq \boldsymbol{i}_Q^M(E, \tau)$. Note that QN is a parabolic subgroup of G with Levi L and unipotent radical RN. Now, Theorem 14.3(e) shows $\boldsymbol{i}_P^G(W, \sigma) \subseteq \boldsymbol{i}_P^G \, \boldsymbol{i}_Q^M(E, \tau) \cong \boldsymbol{i}_{QN}^G(E, \tau)$. It suffices to show that $\boldsymbol{i}_{QN}^G(E, \tau)$ has finite length, and hence we may assume from the start that (W, σ) is irreducible and cuspidal.

We prove the assertion by descending induction on r. If $r = n$, then $\underline{n} = (1, 1, \ldots, 1)$ so that $G = T$, and there is nothing to show. Assume now $r < n$. Denote $P_1, \ldots, P_{n-r}$ the maximal parabolic subgroups of G with Levi subgroups $M_1, \ldots, M_{n-r}$, respectively; note that each M_j is of the form $M_{\underline{n}'}$, where $\underline{n}' = (n_1, \ldots, n_{a-1}, m, n_a - m, n_{a+1}, \ldots, n_r)$ for some $1 \leqslant a \leqslant r$ and $1 \leqslant m < n_a$. In particular, the induction hypothesis is applicable for each M_j. By the Geometrical Lemma 18.2, and because (W, σ) is cuspidal, $\boldsymbol{r}_{P_j}^G \, \boldsymbol{i}_P^G W$ has a finite filtration with graded pieces of the form

$$\boldsymbol{i}_{w^{-1}Pw \cap M_j}^{M_j}(W, w_*^{-1}\sigma), \tag{3.15}$$

where $w \in \{ w \in \mathcal{W}^{P, P_j} \mid M \subseteq w M_j w^{-1} \}$. By the induction hypothesis, each representation (3.15) has finite length. It follows that $\boldsymbol{r}_{P_j}^G \, \boldsymbol{i}_P^G(W, \sigma)$ has finite length, say l_j, for all $j = 1, \ldots, n - r$.

Write $(V, \pi) \coloneqq \boldsymbol{i}_P^G(W, \sigma)$, and let $\{0\} = V_0 \subsetneq V_1 \subsetneq \cdots \subsetneq V_l = V$ be a finite filtration. As each $\boldsymbol{r}_{P_j}^G$ is exact (Theorem 14.3(b)), we obtain for all j a filtration

$$\{0\} \subseteq \boldsymbol{r}_{P_j}^G V_1 \subseteq \boldsymbol{r}_{P_j}^G V_2 \subseteq \cdots \subseteq \boldsymbol{r}_{P_j}^G V_l = \boldsymbol{r}_{P_j}^G \, \boldsymbol{i}_P^G W.$$

For each $1 \leqslant i \leqslant l$, Corollary 17.10 shows that V_i / V_{i-1} is not cuspidal; hence, there exists j such that $\boldsymbol{r}_{P_j}^G(V_i) / \boldsymbol{r}_{P_j}^G(V_{i-1}) \cong \boldsymbol{r}_{P_j}^G(V_i / V_{i-1}) \neq \{0\}$. It follows that $l \leqslant l_1 + \cdots + l_{n-r}$, hence $\boldsymbol{i}_P^G(W, \sigma)$ has finite length. □

Theorem 19.2 (Jacquet's Lemma)**.** *Let $P = MN$ be a parabolic subgroup of G and let $(V, \pi) \in \operatorname{Rep}(G)$ be admissible. For every $m \geqslant 1$ the projection $\operatorname{pr}_N \colon V \twoheadrightarrow J_N(V)$ induces a surjection $V^{K_m} \twoheadrightarrow J_N(V)^{K_m \cap M}$.*

Proof. Fix $\lambda \in \Lambda^{++}(M,G)$ (see Notation 12.14) and put $K_m^- = K_m \cap \overline{N}$, $K_m^0 = K_m \cap M$ and $K_m^+ = K_m \cap N$, see Proposition 12.15. It is clear that $\mathrm{pr}_N(V^{K_m}) \subseteq J_N(V)^{K_m^0}$. For each $l \geqslant 0$, we have a decomposition

$$H_l := K_m \cap \lambda^l K_m \lambda^{-l} = K_m^- K_m^0 (\lambda^l K_m^+ \lambda^{-l}).$$

Since also $K_m = K_m^+ K_m^0 K_m^-$, the inclusion $K_m^+/\lambda^l K_m^+ \lambda^{-l} \hookrightarrow K_m/H_l$ is bijective. Now, for all $v \in V^{\lambda^{-l} H_l \lambda^l}$, we have $\pi(e_{K_m})\pi(\lambda^l)v = \frac{1}{[K_m:H_l]} \sum_{u \in K_m^+/\lambda^l K_m^+ \lambda^{-l}} \pi(u\lambda^l)v \in V^{K_m}$, and hence, since N acts trivially on $J_N(V)$,

$$\pi_N(\lambda)^l \,\mathrm{pr}_N(v) = \mathrm{pr}_N\big(\pi(e_{K_m})\pi(\lambda^l)v\big) \in \mathrm{pr}_N(V^{K_m}). \tag{3.16}$$

As $V^{K_m} \subseteq V^{\lambda^{-l} H_l \lambda^l}$, it follows that $\pi_N(\lambda)^l \,\mathrm{pr}_N(V^{K_m}) \subseteq \mathrm{pr}_N(V^{K_m})$. Since V is admissible, $\mathrm{pr}_N(V^{K_m})$ is finite dimensional, and hence $\pi_N(\lambda)$ is invertible on $\mathrm{pr}_N(V^{K_m})$. We deduce

$$\pi_N(\lambda)^l \,\mathrm{pr}_N\big(V^{K_m}\big) = \mathrm{pr}_N\big(V^{K_m}\big) \quad \text{for all } l \in \mathbb{Z}. \tag{3.17}$$

Let $\overline{v} \in J_N(V)^{K_m^0} = J_N(V)^{K_m^0 K_m^+}$. Since $(\_)^{K_m^0 K_m^+}$ is exact by Lemma 5.8, we find $v \in V^{K_m^0 K_m^+}$ with $\mathrm{pr}_N(v) = \overline{v}$. Using Proposition 12.15, we see that v is fixed by $\lambda^{-l} K_m^- \lambda^l$ for some $l \geqslant 0$, and hence $v \in V^{\lambda^{-l} H_l \lambda^l}$. By (3.16), we find

$$\pi_N(\lambda)^l \overline{v} = \mathrm{pr}_N\big(\pi(e_{K_m})\pi(\lambda^l)v\big) \in \mathrm{pr}_N(V^{K_m}),$$

and from (3.17) we deduce $\overline{v} \in \pi_N(\lambda)^{-l} \,\mathrm{pr}_N(V^{K_m}) = \mathrm{pr}_N(V^{K_m})$. □

Corollary 19.3. *Let $(V,\pi) \in \mathrm{Rep}(G)$ be admissible. For each parabolic subgroup $P = MN$ of G, the representation $\boldsymbol{r}_P^G(V,\pi) \in \mathrm{Rep}(M)$ is admissible.*

Proof. Immediate from Theorem 19.2. □

Lemma 19.4. *Let $P = MN$ be a standard parabolic subgroup of G. Let $(V,\pi) \in \mathrm{Rep}(G)$ and suppose that V is generated by V^{K_m} for some $m \geqslant 1$. Then $J_N(V)$ is generated by $J_N(V)^{K_m \cap M}$ as an M-representation.*

Proof. By the Iwasawa decomposition 12.7 we have $G = PK$. As K_m is a normal subgroup in K, we have $\pi(k)V^{K_m} = V^{K_m}$ for all $k \in K$, and hence it follows that V^{K_m} generates V as a P-representation. The image of V^{K_m} under the P-equivariant surjection $V \to J_N(V)$ lies in $J_N(V)^{K_m \cap M}$. As N acts trivially on $J_N(V)$ and $P/N \cong M$, the claim follows. □

Proposition 19.5. *Suppose $(V,\pi) \in \mathrm{Rep}(G)$ is generated by V^{K_m} for some $m \geqslant 1$. Then every subquotient (W,σ) of (V,π) is generated by W^{K_m}.*

Proof. Step 1: We show $W^{K_m} \neq \{0\}$ for all non-zero subquotients (W,σ) of (V,π). There exists a (not necessarily proper) parabolic subgroup $P = MN$ of G such that $\boldsymbol{r}_P^G(W,\sigma)$ is cuspidal. By Lemma 19.4, $\boldsymbol{r}_P^G(V,\pi)$ is generated by its $K_m \cap M$-fixed vectors. Since $\boldsymbol{r}_P^G$ is exact by Theorem 14.3(b), $\boldsymbol{r}_P^G(W,\sigma)$ is a cuspidal subquotient of $\boldsymbol{r}_P^G(V,\pi)$. Hence, Corollary 17.9 shows $\boldsymbol{r}_P^G(W,\sigma)^{K_m \cap M} \neq \{0\}$. By Jacquet's Lemma 19.2, the map $W^{K_m} \twoheadrightarrow J_N(W)^{K_m \cap M} \neq \{0\}$ is surjective. Hence $W^{K_m} \neq \{0\}$.

Step 2: Let (W,σ) be a subquotient of (V,π), and let $W' \subseteq W$ be the subrepresentation generated by W^{K_m}. By construction, we have $(W')^{K_m} = W^{K_m}$. As K_m is exact by Lemma 5.8, we deduce $(W/W')^{K_m} = \{0\}$. As W/W' is a subquotient of (V,π), Step 1 implies $W/W' = \{0\}$, that is, $W' = W$. □

Theorem 19.6 (Howe)**.** *Let $(V,\pi) \in \mathrm{Rep}(G)$. Then (V,π) is finitely generated and admissible if and only if (V,π) has finite length.*

Proof. Suppose (V,π) has finite length, say l. We show by induction on l that (V,π) is finitely generated and admissible. If $l = 1$, then (V,π) is generated by any non-zero vector and is admissible by Theorem 15.4. If $l > 1$, we find a G-invariant subspace $W \subseteq V$ such that W and V/W have length $< l$. By induction hypothesis, W and V/W are finitely generated and admissible. From the short exact sequence $0 \to W \to V \to V/W \to 0$ it follows easily that V is finitely generated. For each open subgroup $H \subseteq G$ we have an exact sequence

$$0 \longrightarrow W^H \longrightarrow V^H \longrightarrow (V/W)^H,$$

where W^H and $(V/W)^H$ are finite dimensional by induction hypothesis so that also V^H is finite dimensional. Therefore, (V,π) is admissible.

Suppose now that (V,π) is admissible and finitely generated by, say, $v_1,\dots,v_l$. Fix $m \geqslant 1$ such that $v_1,\dots,v_l \in V^{K_m}$. Let now

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq \cdots \subsetneq V_s = V$$

be a filtration by G-invariant subspaces. From Proposition 19.5 (and Lemma 5.8) we deduce

$$\{0\} \subsetneq V_1^{K_m} \subsetneq V_2^{K_m} \subsetneq \cdots \subsetneq V_s^{K_m} = V^{K_m}$$

(because $V_i^{K_m}/V_{i-1}^{K_m} = (V_i/V_{i-1})^{K_m} \neq \{0\}$ for all i). Since, V is admissible, we have $s \leqslant \dim V^{K_m} < \infty$. Hence, V has finite length. □

Corollary 19.7. *Suppose $(V,\pi) \in \mathrm{Rep}(G)$ has finite length. Let $P = MN$ be a parabolic subgroup of G. Then $\boldsymbol{r}_P^G(V,\pi) \in \mathrm{Rep}(M)$ has finite length.*

Proof. By Theorem 19.6, we have to show that if (V,π) is admissible and finitely generated, then $\boldsymbol{r}_P^G(V,\pi)$ is admissible and finitely generated. But this is Corollary 19.3 and Theorem 14.3(c). □

§20. Cuspidal Data

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1,\dots,n_r)$ of n. We have obtained in Theorem 17.8 a decomposition

$$\mathrm{Rep}(G) = \mathrm{Rep}(G)_{\mathrm{cusp}} \times \mathrm{Rep}(G)_{\mathrm{ind}}.$$

Our aim in this section is to describe $\mathrm{Rep}(G)_{\mathrm{ind}}$ in terms of cuspidal representations of Levi subgroups of G.

Definition 20.1. A *cuspidal datum* is a pair (M,ρ), where $M \subseteq G$ is a Levi subgroup and $\rho \in \mathbf{Irr}_{\mathrm{cusp}}(M)$. We say two cuspidal data (M,ρ) and (M',ρ') are *associated*, and write $(M,\rho) \sim (M',\rho')$ if there exists $g \in G$ such that

$$gMg^{-1} = M' \qquad \text{and} \qquad g_*\rho \cong \rho' \quad \text{in } \mathrm{Rep}(M').$$

The relation $\sim$ is an equivalence relation. We denote $(M,\rho)_G$ the equivalence class of (M,ρ) and put

$$\Omega(G) := \text{ set of equivalence classes } (M,\rho)_G \text{ of cuspidal data.}$$

Lemma 20.2. *Let $\pi \in \mathbf{Irr}(G)$. There exists a standard parabolic subgroup $P = MN$ of G and a cuspidal datum (M, ρ) such that $\pi \hookrightarrow \boldsymbol{i}_P^G\, \rho$.*

Proof. This is Lemma 15.2. □

Theorem 20.3. *Let $P = MN$, $Q = LR$, and $Q' = L'R'$ be parabolic subgroups of G, and let (L, σ) and (L', σ') be cuspidal data. Fix $(V, \pi) \in \mathrm{Rep}(G)$.*

(a) Let $\rho \in \mathrm{Rep}(M)$ be a cuspidal representation. If (V, π) is a subquotient of $\boldsymbol{i}_P^G\, \rho$ and $\sigma \in \mathrm{JH}(\boldsymbol{r}_Q^G\, \pi)$, then σ is a subquotient of $w_*\rho$ for some $w \in \mathcal{W}_G$ with $wMw^{-1} = L$. In particular, if ρ is irreducible, then $(L, \sigma) \sim (M, \rho)$.

(b) Suppose (V, π) is irreducible. If $\sigma \in \mathrm{JH}(\boldsymbol{r}_Q^G\, \pi)$ and $\sigma' \in \mathrm{JH}(\boldsymbol{r}_{Q'}^G\, \pi)$, then $(L, \sigma) \sim (L', \sigma')$. In particular, there exists a unique $(L, \sigma)_G \in \Omega(G)$ such that π is a subrepresentation/subquotient of $\boldsymbol{i}_Q^G\, \sigma$ for some parabolic subgroup $Q \subseteq G$ with Levi L.

Proof. We first argue that (b) follows from (a). By Lemma 20.2 there exists a standard parabolic subgroup $P = MN$ of G and a cuspidal datum (M, ρ) such that $\pi \hookrightarrow \boldsymbol{i}_P^G\, \rho$. The hypotheses together with (a) imply $(L, \sigma) \sim (M, \rho) \sim (L', \sigma')$.

It remains to prove (a). Note that $\pi(g)$ induces an isomorphism $g_*\, \boldsymbol{r}_Q^G\, \pi \xrightarrow{\cong} \boldsymbol{r}_{gQg^{-1}}^G\, \pi$. Replacing (L, σ) with $(gLg^{-1}, g_*\sigma)$ if necessary, we may assume that Q is standard. As in the proof of Theorem 19.1 we may assume that P is standard. Since $\boldsymbol{r}_Q^G$ is exact by Theorem 14.3(b), we have $\sigma \in \mathrm{JH}(\boldsymbol{r}_Q^G\, \pi) \subseteq \mathrm{JH}\big(\boldsymbol{r}_Q^G\, \boldsymbol{i}_P^G\, \rho\big)$. Put $(E, \tau) = \boldsymbol{r}_Q^G\, \boldsymbol{i}_P^G\, \rho$, so that $\sigma \in \mathrm{JH}(E_{\mathrm{cusp}})$. By the Geometrical Lemma 18.2, σ is a subquotient of

$$\big(\boldsymbol{i}_{w^{-1}Pw\cap L}^L\, w_*^{-1}\, \boldsymbol{r}_{M\cap wQw^{-1}}^M\, \rho\big)_{\mathrm{cusp}} \tag{3.18}$$

for some $w \in \mathcal{W}^{P,Q}$. Since ρ is cuspidal, we have $M \cap wQw^{-1} = M$ and hence $M \subseteq wLw^{-1}$ (see Lemma 18.1). Corollary 17.10 shows $L = w^{-1}Pw \cap L$ and hence $L \subseteq w^{-1}Mw$. Together we obtain $w^{-1}Mw = L$, and (3.18) simplifies to $w_*^{-1}\rho$, which is what we wanted to show. □

Exercise. $(V, \pi) \in \mathbf{Irr}(G)$ is called *supercuspidal* if for all proper parabolic subgroups $P = MN$ of G and all $(W, \sigma) \in \mathrm{Rep}(M)$ we have $\pi \notin \mathrm{JH}(\boldsymbol{i}_P^G\, \sigma)$. Show that $(V, \pi) \in \mathbf{Irr}(G)$ is supercuspidal if and only if it is cuspidal.

Definition 20.4. Theorem 20.3 supplies a well-defined map

$$\begin{aligned} \mathbf{Sc}\colon \mathbf{Irr}(G) &\longrightarrow \Omega(G), \\ \pi &\longmapsto (M, \rho)_G \text{ with } \rho \in \mathrm{JH}(\boldsymbol{r}_P^G\, \pi) \text{ for} \\ &\qquad \text{some parabolic } P \subseteq G \text{ with Levi } M, \end{aligned}$$

called the *(super)cuspidal support*.

Note that Theorem 20.3 also shows that $\mathbf{Sc}(\pi) = (M, \rho)_G$ if and only if $\pi \in \mathrm{JH}(\boldsymbol{i}_P^G\, \rho)$ for some parabolic P with Levi M.

The definition suggests that $\Omega(G)$ plays an important role in describing the category $\mathrm{Rep}(G)$. We therefore need to study how strong the relation is between $\mathbf{Irr}(G)$ and $\Omega(G)$ as exhibited by the map $\mathbf{Sc}$. We then show that $\Omega(G)$ is naturally a disjoint union of $\mathbb{C}$-varieties.

Lemma 20.5 (Obsolete)**.** *Let $(V,\pi) \in \operatorname{Rep}(G)$ have finite length, and let $(W,\sigma) \in \mathbf{Irr}_{\mathrm{cusp}}(G)$ such that $W \in \operatorname{JH}(V)$. Then there exists a G-equivariant surjection $V \twoheadrightarrow W$.*

Proof. By Corollary 11.14, (W,σ) is (projective and) injective in $\operatorname{Rep}(G^0)$. Hence, if $V' \subseteq V$ is a G-invariant subspace and $V' \twoheadrightarrow W$ a surjection, then the restriction map

$$X \coloneqq \operatorname{Hom}_{G^0}(V,W) \twoheadrightarrow \operatorname{Hom}_{G^0}(V',W) =: Y$$

is surjective. From $Y \neq \{0\}$ we deduce $X \neq \{0\}$. By Proposition 17.3, $(V,\pi_{|G^0})$ and $(W,\sigma_{|G^0})$ have finite length. By Lemma 16.2, X is finite dimensional. Define a group action $\tau\colon G \to \operatorname{Aut}_{\mathbb{C}}(X)$ via $\tau(g)f \coloneqq \sigma(g)\circ f \circ \pi(g^{-1})$, where $g \in G$ and $f \in X$. We need to show $X^G \neq \{0\}$.

As G^0 acts trivially on X and Y, the G-action factors through the abelian group $G/G^0 = \Lambda(G) \cong \mathbb{Z}^r$. For each $\Lambda(G)$-representation (E,κ) and all $\chi \in \mathcal{X}(G)$ we denote

$$E_\chi \coloneqq \left\{ v \in E \,\middle|\, \text{for all } g \in \Lambda(G) \text{ there exists } l \geqslant 0 \text{ such that } (\kappa(g)-\chi(g))^l v = 0 \right\}$$

the generalized eigenspace of E. If E is finite dimensional, we have a Jordan decomposition $E = \bigoplus_{\chi\in\mathcal{X}(G)} E_\chi$, where $E_\chi \neq \{0\}$ for only finitely many $\chi \in \mathcal{X}(G)$. The surjectivity of φ implies $\varphi(X_\chi) = Y_\chi$, for all $\chi \in \mathcal{X}(G)$. In particular, $\varphi(X_{\mathbf{1}}) = Y_{\mathbf{1}} \neq \{0\}$, where $\mathbf{1} \in \mathcal{X}(G)$ denotes the trivial character. This shows $X_{\mathbf{1}} \neq \{0\}$ and hence also $X^G \neq \{0\}$. □

Proposition 20.6. *The map $\mathbf{Sc}\colon \mathbf{Irr}(G) \to \Omega(G)$ is surjective with finite fibers.*

Proof. We prove surjectivity. Let $P = MN$ be a parabolic subgroup of G and let (M,ρ) be a cuspidal datum. Let π be an irreducible subquotient of $\boldsymbol{i}_P^G\,\rho$. Then Theorem 20.3 shows $\mathbf{Sc}(\pi) = (M,\rho)_G$.

We now prove that every fiber is finite. Fix $(M,\rho)_G \in \Omega(G)$, and let $(V,\pi) \in \mathbf{Sc}^{-1}((M,\rho)_G)$. By Lemma 20.2 there exists a cuspidal datum (M',ρ') and a parabolic subgroup $P' = M'N'$ of G such that $\pi \subseteq \boldsymbol{i}_{P'}^G\,\rho'$. Then Theorem 20.3 shows $(M',\rho') \sim (M,\rho)$, *i.e.*, there exists $g \in G$ with $M = gM'g^{-1}$ and $\rho \cong g_*\rho'$. Put $P \coloneqq gP'g^{-1}$. We have isomorphims $\boldsymbol{i}_{P'}^G\,\rho' \cong g_*\,\boldsymbol{i}_{P'}^G\,\rho' \cong \boldsymbol{i}_P^G\,\rho$, where the first map is induced by the action of g^{-1} and the second map is given by $f \mapsto [\gamma \mapsto f(g^{-1}\gamma)]$. Observe that P lies in the set $\mathcal{P}(M)$ of all parabolic subgroups of G with Levi M. We deduce that the cardinality of $\mathbf{Sc}^{-1}\big((M,\rho)_G\big)$ is bounded above by $\sum_{P\in\mathcal{P}(M)} \ell(\boldsymbol{i}_P^G\,\rho)$, which is finite because $\mathcal{P}(M)$ is finite by Exercise 12.12 and each $\boldsymbol{i}_P^G\,\rho$ has finite length by Theorem 19.1. □

Exercise. Let M be a Levi subgroup in G and recall the set $\mathcal{P}(M)$ of parabolic subgroups of G with Levi M. Fix $(W,\rho) \in \mathbf{Irr}_{\mathrm{cusp}}(M)$ and let $P \in \mathcal{P}(M)$.

(a) Show that for every $\pi \in \operatorname{JH}(\boldsymbol{i}_P^G\,\rho)$ there exists $Q \in \mathcal{P}(M)$ such that $\pi \subseteq \boldsymbol{i}_Q^G\,\rho$.

(b) Show that $\operatorname{Hom}_G(\boldsymbol{i}_P^G\,\rho, \boldsymbol{i}_Q^G\,\rho) \neq \{0\}$ for all $Q \in \mathcal{P}(M)$.

Definition 20.7. We say two cuspidal data (M,ρ), (M',ρ') are *inertially equivalent*, written $(M,\rho) \simeq (M',\rho')$, if there exist $g \in G$ and $\chi \in \mathcal{X}(M')$ such that $gMg^{-1} = M'$ and $\rho' \cong \chi \otimes g_*\rho$.

We denote $[M,\rho]_G$ the inertial equivalence class of the cuspidal datum (M,ρ) and put

$$\mathcal{B}(G) \coloneqq \quad \text{set of inertial equivalence classes } [M,\rho]_G \text{ of cuspidal data.}$$

Observe that $(M,\rho) \sim (M',\rho')$ implies $(M,\rho) \simeq (M',\rho')$, and hence we have a natural surjective map

$$\Upsilon\colon \Omega(G) \longrightarrow\!\!\!\!\!\rightarrow \mathcal{B}(G).$$

Proposition 20.8. *The set $\Omega(G)$ of equivalence classes of cuspidal data is the disjoint union of $\mathbb{C}$-varieties, and the fibers of Υ are the connected components of $\Omega(G)$.*

Proof. Let M be a Levi subgroup in G and consider the group $\mathcal{W}(M) := N_G(M)/M$. Since $\pi(g)\colon g_*\rho \xrightarrow{\cong} \rho$ is an M-equivariant isomorphism for each $g \in M$ and $\rho \in \mathbf{Irr}_{\mathrm{cusp}}(M)$, the action of $N_G(M)$ on $\mathbf{Irr}_{\mathrm{cusp}}(M)$ factors through $\mathcal{W}(M)$. Note that by Exercise 12.12(d) the group $\mathcal{W}(M)$ is finite. Recall the action of $\mathcal{X}(M) = \mathrm{Hom}_{\mathrm{grp}}(M/M^0, \mathbb{C}^\times)$ on $\mathbf{Irr}_{\mathrm{cusp}}(M)$ given by $\chi \cdot \rho := \chi \otimes \rho$ for all $\chi \in \mathcal{X}(M)$ and $\rho \in \mathbf{Irr}_{\mathrm{cusp}}(M)$. The orbits for this action are by definition the cuspidal components. Denote I_M the set of cuspidal components so that

$$\mathbf{Irr}_{\mathrm{cusp}}(M) = \bigsqcup_{D \in I_M} D.$$

We now investigate the action of $\mathcal{W}(M)$ on $\mathbf{Irr}_{\mathrm{cusp}}(M)$. Let $D \in I_M$ and $\rho \in D$. Since $w_*(\chi \otimes \rho) \cong w_*\chi \otimes w_*\rho$ for all $w \in \mathcal{W}(M)$ and $\chi \in \mathcal{X}(M)$, it follows that wD is a cuspidal component. Therefore, $\mathcal{W}(M)$ acts on I_M. Let $J_M \subseteq I_M$ be a complete set of representatives for the orbit space $I_M/\mathcal{W}(M)$. For each $D \in I_M$ we denote $\mathcal{W}(D) := \{w \in \mathcal{W}(M) \mid wD = D\}$ the stabilizer of D in $\mathcal{W}(M)$. Then $\mathcal{W}(D)$ acts on D and we have a bijection

$$\mathbf{Irr}_{\mathrm{cusp}}(M)/\mathcal{W}(M) \cong \bigsqcup_{D \in J_M} D/\mathcal{W}(D).$$

Each $D \in J_M$ is a connected $\mathbb{C}$-variety (Definition 17.5), hence so is the quotient $D/\mathcal{W}(D)$ by a finite group (Proposition 17.6). It follows that $\mathbf{Irr}_{\mathrm{cusp}}(M)/\mathcal{W}(M)$ is a disjoint union of $\mathbb{C}$-varieties.

Let now $M_1, \ldots, M_l$ be a complete set of representatives for the standard Levi subgroups of G up to conjugation; then every Levi subgroup of G is conjugate to precisely one of the M_i. The above discussion shows that the bijection

$$\Omega(G) \cong \bigsqcup_{i=1}^{l} \mathbf{Irr}_{\mathrm{cusp}}(M_i)/\mathcal{W}(M_i) \cong \bigsqcup_{i=1}^{l} \bigsqcup_{D \in J_{M_i}} D/\mathcal{W}(D)$$

exhibits $\Omega(G)$ as a disjoint union of the connected $\mathbb{C}$-varieties $D/\mathcal{W}(D)$.

Finally, let (M, ρ) be a cuspidal datum and denote D the cuspidal component containing ρ. It is clear from the definition that $\Upsilon^{-1}\big([M,\rho]_G\big) = D/\mathcal{W}(D)$, which proves the last assertion. $\square$

Definition 20.9. The function $\mathbf{Si}\colon \mathbf{Irr}(G) \to \mathcal{B}(G)$, defined by the commutativity of the diagram

$$\begin{array}{ccc} \mathbf{Irr}(G) & \xrightarrow{\mathbf{Sc}} & \Omega(G) \\ & \searrow_{\mathbf{Si}} & \downarrow \Upsilon \\ & & \mathcal{B}(G), \end{array}$$

is called the *inertial support*. Let $\mathfrak{s} \in \mathcal{B}(G)$ and let $\Omega = \Upsilon^{-1}(\mathfrak{s}) \subseteq \Omega(G)$ be the corresponding connected component. We put

$$\mathbf{Irr}_{\mathfrak{s}}(G) := \mathbf{Irr}_{\Omega}(G) := \mathbf{Si}^{-1}(\mathfrak{s}).$$

§21. The Bernstein Decomposition Theorem

Recall $G = M_{\underline{n}}$ for some partition $\underline{n} = (n_1, \dots, n_r)$ of n. We prove in this section the main result of this lecture course.

We consider the category

$$\mathrm{Cusp}(G) := \prod_{\substack{P=MN \\ \text{standard parabolic}}} \mathrm{Rep}(M)_{\mathrm{cusp}}.$$

The objects are tuples $(W_M, \rho_M)_P$, where each (W_M, ρ_M) is a cuspidal M-representation, and for $(W_M, \rho_M)_P, (E_M, \sigma_M)_P \in \mathrm{Cusp}(G)$ we put

$$\mathrm{Hom}_{\mathrm{Cusp}(G)}\big((W_M, \rho_M)_P, (E_M, \sigma_M)_P\big) := \prod_P \mathrm{Hom}_M\big((W_M, \rho_M), (E_M, \sigma_M)\big).$$

Note that $\mathrm{Cusp}(G)$ is an abelian category, because kernels and cokernels of morphisms are computed componentwise. We also consider two functors

$$\begin{aligned} \mathrm{R}\colon \mathrm{Rep}(G) &\rightleftarrows \mathrm{Cusp}(G) : \mathrm{I}, \\ (V, \pi) &\longmapsto \big(\boldsymbol{r}_P^G(V, \pi)_{\mathrm{cusp}}\big)_P, \\ \textstyle\bigoplus_P \boldsymbol{i}_P^G(W_M, \rho_M) &\longleftarrow (W_M, \rho_M)_P. \end{aligned}$$

Lemma 21.1. *(a) For all $(V, \pi) \in \mathrm{Rep}(G)$ and $(W_M, \rho_M)_P \in \mathrm{Cusp}(G)$ we have a natural isomorphism*

$$\mathrm{Hom}_{\mathrm{Cusp}(G)}\big(\mathrm{R}(V, \pi), (W_M, \rho_M)_P\big) \cong \mathrm{Hom}_G\big((V, \pi), \mathrm{I}(W_M, \rho_M)_P\big).$$

In other words, R *is left adjoint to* I.

(b) The functor R *is exact and* faithful*, that is, for all $(V, \pi), (V', \pi') \in \mathrm{Rep}(G)$ the induced map*

$$\mathrm{Hom}_G\big((V, \pi), (V', \pi')\big) \longrightarrow \mathrm{Hom}_{\mathrm{Cusp}(G)}\big(\mathrm{R}(V, \pi), \mathrm{R}(V', \pi')\big)$$

is injective. If $(V, \pi) \in \mathrm{Rep}(G)$ is finitely generated, then each component of $\mathrm{R}(V, \pi)$ is finitely generated.

(c) For all $(V, \pi) \in \mathrm{Rep}(G)$, the map

$$\eta_V \colon (V, \pi) \longrightarrow \mathrm{IR}(V, \pi)$$

corresponding to $\mathrm{id}_{\mathrm{R}(V,\pi)}$ under the bijection in (a) is injective.

Proof. In (a), we compute

$$\begin{aligned}
\mathrm{Hom}_{\mathrm{Cusp}(G)}\big(\mathrm{R}\pi,(\rho_M)_P\big) &= \prod_P \mathrm{Hom}_M\big((\boldsymbol{r}_P^G\,\pi)_{\mathrm{cusp}},\rho_M\big)\\
&\cong \prod_P \mathrm{Hom}_M\big(\boldsymbol{r}_P^G\,\pi,\rho_M\big) && \text{(Theorem 17.8)}\\
&\cong \prod_P \mathrm{Hom}_G\big(\pi,\boldsymbol{i}_P^G\,\rho_M\big) && \text{(Theorem 14.3(a))}\\
&\cong \mathrm{Hom}_G\Big(\pi,\bigoplus_P \boldsymbol{i}_P^G\,\rho_M\Big)\\
&= \mathrm{Hom}_G\big(\pi,\mathrm{I}(\rho_M)_P\big),
\end{aligned}$$

where for the first isomorphism we argue as follows: By Theorem 17.8, we have a decomposition $\boldsymbol{r}_P^G\,\pi = (\boldsymbol{r}_P^G\,\pi)_{\mathrm{cusp}} \oplus (\boldsymbol{r}_P^G\,\pi)_{\mathrm{ind}}$, where no irreducible subquotient of $(\boldsymbol{r}_P^G\,\pi)_{\mathrm{ind}}$ is cuspidal. For each M-equivariant map $f\colon \boldsymbol{r}_P^G\,\pi \to \rho_M$ we thus have $f\big((\boldsymbol{r}_P^G\,\pi)_{\mathrm{ind}}\big) = \{0\}$ by Lemma 16.5(b) and because ρ_M is cuspidal. Hence, f is uniquely determined by its restriction to $(\boldsymbol{r}_P^G\,\pi)_{\mathrm{cusp}}$. The last isomorphism holds, since the direct sum is finite.

We now prove (b). By Theorem 14.3(b) the functors $\boldsymbol{r}_P^G$ are exact. From Theorem 17.8 it also follows that the functor $(W,\rho) \mapsto (W_{\mathrm{cusp}},\rho_{\mathrm{cusp}})$ is exact. Hence R is exact. If $(V,\pi) \in \mathrm{Rep}(G)$ is finitely generated, then $\boldsymbol{r}_P^G(V,\pi)$ is finitely generated by Theorem 14.3(c). But then also its quotient $(\boldsymbol{r}_P^G(V,\pi))_{\mathrm{cusp}}$ (Theorem 17.8) is finitely generated. It remains to prove that R is faithful. We first show that $(V,\pi) \neq \{0\}$ implies $\mathrm{R}(V,\pi) \neq \{0\}$. But this is clear: Let $P = MN$ be a minimal standard parabolic subgroup such that $\boldsymbol{r}_P^G(V,\pi) \neq \{0\}$. Then $\boldsymbol{r}_P^G(V,\pi)$ is cuspidal, and hence the P-component of $\mathrm{R}(V,\pi)$ is non-zero. Let now $f\colon (V,\pi) \to (V',\pi')$ be a non-zero G-equivariant map. We have to show that $\mathrm{R}(f)\colon \mathrm{R}(V,\pi) \to \mathrm{R}(V',\pi')$ is non-zero. Since R is exact, we have

$$\mathrm{R}(V)/\operatorname{Ker}\mathrm{R}(f) \cong \mathrm{R}(V)/\mathrm{R}(\operatorname{Ker}(f)) \cong \mathrm{R}\big(V/\operatorname{Ker}(f)\big) \neq \{0\},$$

and this shows $\mathrm{R}(f) \neq 0$.

For part (c), let $(V,\pi) \in \mathrm{Rep}(G)$ and denote $\iota\colon \operatorname{Ker}\eta_V \hookrightarrow V$ the inclusion of the kernel of η_V into V. Since the isomorphism in (a) is natural, we have a commutative diagram

$$\begin{array}{ccc}
\mathrm{Hom}_G\big(V,\mathrm{IR}(V)\big) & \xrightarrow{\ \cong\ } & \mathrm{Hom}_{\mathrm{Cusp}(G)}\big(\mathrm{R}(V),\mathrm{R}(V)\big)\\
{\scriptstyle\circ\iota}\big\downarrow & & \big\downarrow{\scriptstyle\circ\mathrm{R}(\iota)}\\
\mathrm{Hom}_G\big(\operatorname{Ker}\eta_V,\mathrm{IR}(V)\big) & \xrightarrow[\cong]{} & \mathrm{Hom}_{\mathrm{Cusp}(G)}\big(\mathrm{R}(\operatorname{Ker}\eta_V),\mathrm{R}(V)\big).
\end{array}$$

From $\eta_V \circ \iota = 0$ we deduce $\mathrm{R}(\iota) = \mathrm{id}_{\mathrm{R}(V)} \circ \mathrm{R}(\iota) = 0$. As R is faithful, we deduce $\iota = 0$ which means $\operatorname{Ker}\eta_V = \{0\}$. Hence, η_V is injective. □

Recall that for every inertial equivalence class $\mathfrak{s} \in \mathcal{B}(G)$ we denote $\mathbf{Irr}_{\mathfrak{s}}(G) = \mathbf{Si}^{-1}(\mathfrak{s})$ the set of all irreducible smooth representations $(V,\pi) \in \mathrm{Rep}(G)$ for which there exists a cuspidal datum (M,ρ) with $\mathfrak{s} = [M,\rho]_G$ such that $\rho \in \mathrm{JH}(\boldsymbol{r}_P^G\,\pi)$ for some parabolic subgroup $P \subseteq G$ with Levi M. We denote

$$\mathrm{Rep}(G)_{\mathfrak{s}} := \mathrm{Rep}(G)_{\mathbf{Irr}_{\mathfrak{s}}}$$

the full subcategory of $\mathrm{Rep}(G)$ consisting of the (V,π) such that $\mathrm{JH}(\pi) \subseteq \mathbf{Irr}_{\mathfrak{s}}$.

Theorem 21.2 (Bernstein Decomposition Theorem)**.** *One has*

$$\mathrm{Rep}(G) = \prod_{\mathfrak{s}\in\mathcal{B}(G)} \mathrm{Rep}(G)_{\mathfrak{s}}.$$

Proof. Step 1: Let $P = MN$ be a parabolic subgroup of G, and let $(W,\sigma) \in \mathrm{Rep}(M)_{\mathrm{cusp}}$. We show

$$\boldsymbol{i}_P^G(W) = \bigoplus_{\mathfrak{s}\in\mathcal{B}(G)} (\boldsymbol{i}_P^G W)_{\mathfrak{s}}.$$

As (W,σ) is cuspidal, Theorem 17.8 shows $W = \bigoplus_D W_D$, where D runs through the cuspidal components of $\mathbf{Irr}_{\mathrm{cusp}}(M)$. The natural G-equivariant homomorphism

$$\bigoplus_D \boldsymbol{i}_P^G(W_D) \xrightarrow{\cong} \boldsymbol{i}_P^G\Big(\bigoplus_D W_D\Big) \tag{3.19}$$

is bijective: Indeed, injectivity is obvious from the definition, so we need to show surjectivity. Let $f \in \boldsymbol{i}_P^G\big(\bigoplus_D W_D\big)$. Let $H \subseteq G$ be a compact open subgroup fixing f. As $P\backslash G$ is compact by the Iwasawa decomposition 12.7, the coset space $P\backslash G/H$ is finite. Let $g_1,\dots,g_l \in G$ be a system of representatives for $P\backslash G/H$. Let $D_1,\dots,D_k$ be cuspidal components such that $f(g_i) \in \bigoplus_{j=1}^k W_{D_j}$ for all $1 \leqslant i \leqslant l$. For each $g \in G$, we find $x \in P$, $h \in H$ and i such that $g = xg_ih$; then

$$f(g) = f(xg_ih) = \delta_P^{1/2}(x)\sigma(x)f(g_i) \in \bigoplus_{j=1}^k W_{D_j},$$

which shows $f \in \boldsymbol{i}_P^G\big(\bigoplus_{j=1}^k W_{D_j}\big) = \bigoplus_{j=1}^k \boldsymbol{i}_P^G(W_{D_j}) \subseteq \bigoplus_D \boldsymbol{i}_P^G(W_D)$. By Theorem 20.3 we have $\boldsymbol{i}_P^G(W_D) \subseteq (\boldsymbol{i}_P^G W)_{\mathfrak{s}}$, where $\mathfrak{s} = [M,\rho]_G$ for some (hence all) $\rho \in D$. But then (3.19) shows that we have an equality, which finishes the proof of Step 1.

Step 2: Proof of the theorem. Let $(V,\pi) \in \mathrm{Rep}(G)$. Lemma 21.1 supplies an injection

$$V \hookrightarrow \mathrm{IR}(V) = \bigoplus_{\substack{P=MN \\ \text{standard parabolic}}} \boldsymbol{i}_P^G\big((\boldsymbol{r}_P^G V)_{\mathrm{cusp}}\big).$$

By Step 1 we have $\mathrm{IR}(V) = \bigoplus_{\mathfrak{s}} (\mathrm{IR}(V))_{\mathfrak{s}}$, and Lemma 16.7 shows $V = \bigoplus_{\mathfrak{s}} V_{\mathfrak{s}}$. □

We give a characterization for the objects in the block $\mathrm{Rep}(G)_{\mathfrak{s}}$, for $\mathfrak{s} \in \mathcal{B}(G)$.

Corollary 21.3. *Let $P = MN$ be a standard parabolic subgroup of G, let $D \subseteq \mathbf{Irr}_{\mathrm{cusp}}(M)$ be a cuspidal component. Fix $\rho \in D$ and put $\mathfrak{s} := [M,\rho]_G$. For $(V,\pi) \in \mathrm{Rep}(G)$, the following assertions are equivalent:*

- *(i)* $(V,\pi) \in \mathrm{Rep}(G)_{\mathfrak{s}}$*;*
- *(ii)* (V,π) *is a subrepresentation of* $\bigoplus_Q \boldsymbol{i}_Q^G(W_Q,\sigma_Q)$*, where the direct sum runs through the standard parabolic subgroups* $Q = LR$ *of* G *such that* $L = gMg^{-1}$ *for some* $g \in G$*, and where* $(W_Q,\sigma_Q) \in \mathrm{Rep}(L)_{g_*D}$*.*
- *(iii)* (V,π) *is a subquotient of a representation as in (ii);*

(iv) $\boldsymbol{r}_P^G(V,\pi) \in \prod_{w\in\mathcal{W}(M)} \mathrm{Rep}(M)_{w_*D}$;

(v) Whenever $Q = LR$ *is a standard parabolic subgroup of* G *and* $D' \subseteq \mathbf{Irr}_{\mathrm{cusp}}(L)$ *is a cuspidal component such that* $(gLg^{-1}, g_*D') \neq (M,D)$ *for all* $g \in G$*, then the component of* $\boldsymbol{r}_Q^G(V,\pi)$ *in* $\mathrm{Rep}(L)_{D'}$ *is zero.*

Proof. Since each $\boldsymbol{i}_Q^G(W_Q,\sigma_Q)$ as in (ii) lies in $\mathrm{Rep}(G)_{\mathfrak{s}}$, and $\mathrm{Rep}(G)_{\mathfrak{s}}$ is closed under subquotients, the implications (ii) $\Longrightarrow$ (iii) $\Longrightarrow$ (i) are clear. In the proof of Theorem 21.2 we have seen that

$$V \subseteq \mathrm{IR}(V) = \bigoplus_{(Q,D')} \boldsymbol{i}_Q^G\big((\boldsymbol{r}_Q^G V)_{D'}\big),$$

where the direct sum runs through the pairs (Q,D'), where $Q = LR$ is a standard parabolic subgroup of G and $D' \subseteq \mathbf{Irr}_{\mathrm{cusp}}(L)$ is a cuspidal component. Since $\mathrm{IR}(V)_{\mathfrak{s}}$ has the form described in (ii), we obtain the implication (i) $\Longrightarrow$ (ii).

Finally, the implications (v) $\Longrightarrow$ (iv) $\Longrightarrow$ (i) $\Longrightarrow$ (v) are clear from the definitions. $\square$

Let now $P = MN$ be a parabolic subgroup of G. A cuspidal datum (L,σ) of M is also a cuspidal datum of G. We obtain maps

$$i_{GM}\colon \Omega(M) \longrightarrow \Omega(G), \quad (L,\sigma)_M \longmapsto (L,\sigma)_G, \qquad i_{GM}\colon \mathcal{B}(M) \longrightarrow \mathcal{B}(G), \quad [L,\sigma]_M \longmapsto [L,\sigma]_G.$$

Corollary 21.4.

(a) Let $\mathfrak{s} \in \mathcal{B}(M)$ *and* $(W,\rho) \in \mathrm{Rep}(M)_{\mathfrak{s}}$*. Then* $\boldsymbol{i}_P^G(W,\rho) \in \mathrm{Rep}(G)_{i_{GM}(\mathfrak{s})}$*.*

(b) Let $\mathfrak{t} \in \mathcal{B}(G)$ *and* $(V,\pi) \in \mathrm{Rep}(G)_{\mathfrak{t}}$*. Then* $\boldsymbol{r}_P^G(V,\pi) \in \prod_{\mathfrak{s}\in i_{GM}^{-1}(\mathfrak{t})} \mathrm{Rep}(M)_{\mathfrak{s}}$*.*

Proof. Part (a) follows from the equivalence (i) $\Longleftrightarrow$ (ii) in Corollary 21.3 and the transitivity of $\boldsymbol{i}_P^G$ (Theorem 14.3(e)), whereas (b) follows from the equivalence (i) $\Longleftrightarrow$ (iv) in Corollary 21.3 and the transitivity of $\boldsymbol{r}_P^G$.

Alternatively, check this directly using Theorem 20.3 (and Theorem 21.2). $\square$

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