TALK 7: DISCRETE ADIC SPACES

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Contents

Notation: We denote (Profin) the category of profinite sets and $(\mathsf{ExtrDisc})$ its full subcategory of extremally disconnected sets.

If A is a condensed ring, we write $\mathrm{Mod}(A)$ for the category of condensed A-modules. We denote $\mathcal{D}(A)$ the (unbounded) derived ∞-category of $\mathrm{Mod}(A)$, and $\mathcal{D}^{\leq 0}(A)$ its full subcategory of connective complexes. We write $\mathrm{R}\underline{\mathrm{Hom}}_A$ for the internal Hom of $\mathcal{D}(A)$.

1. Static Analytic Rings

1.1. In this section, unless specified otherwise, all the rings are *classical* condensed rings, *i.e.*, "underived" rings endowed with the discrete topology. The main objective of this section is to recall the notion of analytic rings in this context and study some of their properties. As the main result, we show that any $\mathbb{Z}$-algebra of finite type comes with a natural analytic structure.

Recall the following definition:

1.2. **Definition.** (a) A *pre-analytic ring* is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M})$, where $\underline{\mathcal{A}}$ is a condensed ring and

$$\mathcal{M} \colon (\mathsf{Profin}) \longrightarrow \mathrm{Mod}(\underline{\mathcal{A}}),$$
$$S \longmapsto \mathcal{A}[S]$$

is a functor from profinite sets to the category of condensed $\underline{\mathcal{A}}$-modules, together with a natural transformation $\underline{\mathcal{A}}[S] \to \mathcal{A}[S]$ of functors $(\mathsf{Profin}) \to \mathrm{Mod}(\underline{\mathcal{A}})$.

(b) A pre-analytic ring $\mathcal{A}$ is called *analytic* if in addition the following properties hold:
   (i) For any $S, T \in (\mathsf{Profin})$ we have $\mathcal{M}(S \sqcup T) = \mathcal{M}(S) \oplus \mathcal{M}(T)$, and the map $\underline{\mathcal{A}} \to \mathcal{A}[*]$ is an isomorphism of $\underline{\mathcal{A}}$-modules.
   (ii) For any complex $C = [\cdots \to C_n \to \cdots \to C_0 \to 0]$ of $\underline{\mathcal{A}}$-modules of the form $C_n = \bigoplus_i \mathcal{A}[S_i]$, for varying profinite sets S_i, the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}\big(\mathcal{A}[S], C\big) \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}\big(\underline{\mathcal{A}}[S], C\big)$$

   is an in the derived category $\mathcal{D}(\underline{\mathcal{A}})$ of $\mathrm{Mod}(\underline{\mathcal{A}})$.

(c) If $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M})$ is an analytic ring, a complex $M \in \mathcal{D}(\underline{\mathcal{A}})$ is called $\mathcal{A}$-*analytic* if for any profinite set S the canonical map

$$\mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\mathcal{A}[S], M) \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}(\underline{\mathcal{A}}[S], M)$$

is an isomorphism in $\mathcal{D}(\underline{\mathcal{A}})$. We denote $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$ the full subcategory consisting of $\mathcal{A}$-analytic complexes.

We will later consider a general notion of analytic structures on animated condensed rings.

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1.3. **Remark.** The full subcategory $\mathcal{D}(\mathcal{A})$ of $\mathcal{D}(\underline{\mathcal{A}})$ is closed under small (co)limits and extensions and is generated by the compact projective objects $\mathcal{A}[S]$, where S runs through the extremally disconnected sets.

The inclusion $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$ admits a left adjoint $- \otimes_{\underline{\mathcal{A}}} \mathcal{A}$ sending $\underline{\mathcal{A}}[S] \mapsto \mathcal{A}[S]$. Moreover, $M \otimes_{\mathcal{A}} N := (M \otimes_{\underline{\mathcal{A}}} N) \otimes_{\underline{\mathcal{A}}} \mathcal{A}$ is the unique symmetric monoidal structure on $\mathcal{D}(\mathcal{A})$ making $- \otimes_{\underline{\mathcal{A}}} \mathcal{A}$ a symmetric monoidal functor. In particular, we have $\mathcal{A}[S] \otimes_{\mathcal{A}} \mathcal{A}[T] \cong \mathcal{A}[S \times T]$, for all profinite sets S, T.

1.4. **Example.** Let A be a $\mathbb{Z}$-algebra of finite type.

(a) We define a pre-analytic ring $A_\square$ with underlying condensed ring A and $A_\square[S] := \varprojlim_i A[S_i]$, where the S_i are finite sets and $S := \varprojlim_i S_i$. Note that there is a canonical natural transformation $A[S] \to A_\square[S]$.

We have seen that, for any profinite set $S = \varprojlim_i S_i$, the abelian group $C(S, \mathbb{Z})$ of continuous maps $S \to \mathbb{Z}$ is free. Since we have isomorphisms of condensed A-modules $A_\square[S] = \varprojlim_i A[S_i] \cong \varprojlim_i \underline{\mathrm{Hom}}(C(S_i, \mathbb{Z}), A) \cong \underline{\mathrm{Hom}}(\varinjlim_i C(S_i, \mathbb{Z}), A) \cong \underline{\mathrm{Hom}}(C(S, \mathbb{Z}), A)$, we deduce an isomorphism

$$A_\square[S] \cong A^I \tag{1.5}$$

of condensed A-modules, where I is some set satisfying $|I| \leq 2^{|S|}$.

(b) Let $A \to B$ be a ring morphism. Then $(B, A)_\square[S] := B \otimes_A A_\square[S]$ defines a pre-analytic ring structure on B.

Our goal in this section is to prove the following theorem.

1.6. **Theorem.** *Let A be a ring of finite type over $\mathbb{Z}$. The pair*

$$A_\square := \big(A, S \mapsto \varprojlim_i A[S_i]\big)$$

is an analytic ring.

We prepare the proof with a sequence of lemmas.

1.7. **Lemma.** *Let $A \to B$ be a morphism between $\mathbb{Z}$-algebras, and assume that A is of finite type.*

(a) If $A \to B$ is finite, then $B_\square = (B, A)_\square$.

(b) If $A_\square$ is analytic, then so is $(B, A)_\square$.

Proof. We first make an observation: given a discrete A-module M and any set I, we have an isomorphism

$$M \overset{\mathrm{L}}{\underset{A}{\otimes}} A^I \xrightarrow{\cong} M \otimes_A A^I. \tag{1.8}$$

Indeed both sides commute with colimits in M, so we may assume that M is finitely generated. In this case, we claim that the canonical map

$$M \overset{\mathrm{L}}{\underset{A}{\otimes}} A^I \longrightarrow M^I \tag{1.9}$$

is an isomorphism in $\mathcal{D}(A)$, which proves the claim since the right-hand side is concentrated in degree 0. But this follows by resolving M by finite free A-modules (using that A is noetherian).

We will use (1.8) *implicitly from now on.*

To prove (a), let S be a profinite set. We have to show that the canonical map $B \otimes_A A_\square[S] \to B_\square[S]$ is an isomorphism in $\mathcal{D}(B)$. But since $B_\square[S] \cong B^I$ by (1.5), this follows from (1.9).

We now prove (b). Let S be a profinite set and $C = [\cdots \to C_n \to \cdots \to C_1 \to C_0]$ be a complex of the form $C_n = \bigoplus_i B \otimes_A A_\square[S_i]$, for varying profinite sets S_i. Note that $B \otimes_A A_\square[S_i]$ is an $A_\square$-analytic A-module: since $A_\square$-analytic A-modules are closed under small colimits, we easily reduce to the case where B is a finitely generated A-module so that $B \otimes_A A_\square[S_i] \cong B^I$ by (1.9), which as a product of discrete (hence $A_\square$-analytic) A-modules is again $A_\square$-analytic. Consequently, C is $A_\square$-analytic.

We now have natural isomorphisms

$$\begin{aligned} \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{D}(B)}\big(B \otimes_A A_\square[S], C\big) &\cong \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{D}(A)}\big(A_\square[S], C\big) \\ &\cong \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{D}(A)}\big(A[S], C\big) \cong \mathrm{R}\underline{\mathrm{Hom}}_{\mathcal{D}(B)}\big(B[S], C\big). \end{aligned}$$

□

1.10. **Remark.** In the situation of Lemma 1.7.(b), the category $\mathcal{D}_\square(B,A) := \mathcal{D}\big((B,A)_\square\big)$ consists precisely of those objects in $\mathcal{D}(B)$ mapping to $\mathcal{D}_\square(A) := \mathcal{D}(A_\square)$ under the forgetful functor.

It follows from (1.5) that for profinite sets S, the A-module $A_\square[S]$ is compact projective. Hence, it follows formally from Remark 1.3 that the full subcategory of compact objects in $\mathcal{D}_\square(A)$ is closed under the tensor product. The analogous statement is true for $\mathcal{D}_\square(B,A)$.

1.11. **Lemma.** *Consider the $A[x]$-algebra $A[x]_\infty := A((x^{-1}))$, which we view as the algebra of "functions near infinity".*

Assume that $A_\square$ is an analytic ring.

(a) The $A[x]$-module Q, defined by the exact sequence

$$0 \longrightarrow A[x] \otimes_A A_\square[S] \longrightarrow A[x]_\square[S] \longrightarrow Q \longrightarrow 0$$

is naturally an $A[x]_\infty$-module, for any profinite set S.

(b) The $A[x]$-module $A[x]_\infty$ is compact in $\mathcal{D}_\square(A[x],A)$.

(c) One has $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, M) = 0$ for any discrete $A[x]$-module M.

Proof. To prove (a), recall that $A_\square[S] \cong A^I$ and $A[x]_\square[S] \cong A[x]^I$ by (1.5). Now notice that we have a pushout diagram

$$\begin{tikzcd} A[x] \otimes_A A^I \arrow[rr] \arrow[d] & & A[x]^I \arrow[d] \\ A((x^{-1})) \otimes_{A[\![x^{-1}]\!]} A[\![x^{-1}]\!]^I \arrow[rr] \arrow[urr, phantom, "\ulcorner"] & & A((x^{-1}))^I. \end{tikzcd}$$

The assertion follows from the observation that Q is also the cokernel of the bottom map.

Let us show (b). Observe that $A[\![y]\!] \cong A^{\mathbb{N}}$ is compact in $\mathcal{D}_\square(A)$ and hence $A[x] \otimes_A A[\![y]\!]$ is compact in $\mathcal{D}_\square(A[x],A)$. Since we have a cofiber sequence

$$A[x] \otimes_A A[\![y]\!] \xrightarrow{\cdot(x\otimes y - 1\otimes 1)} A[x] \otimes_A A[\![y]\!] \longrightarrow A((x^{-1})), \tag{1.12}$$

which is in fact a short exact sequence, it follows that also $A((x^{-1}))$ is compact.

We now prove (c). The resolution (1.12) shows that it suffices to prove that the top map in the diagram

$$\begin{tikzcd} \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}\big(A[x] \otimes_A A[\![y]\!], M\big) \arrow[r, "x\otimes y - 1\otimes 1"] \arrow[d, "\cong"'] & \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}\big(A[x] \otimes_A A[\![y]\!], M\big) \arrow[d, "\cong"] \\ \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[\![y]\!], M\big) \arrow[r, "f\mapsto xf(y\cdot)-f"'] & \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[\![y]\!], M\big) \\ M[y^{-1}]/M \arrow[r, "\psi", "\cdot(xy-1)"'] \arrow[u, "\varphi"] & M[y^{-1}]/M \arrow[u, "\varphi"'] \end{tikzcd}$$

is an isomorphism. We will construct an isomorphism φ which makes the diagram commute. Given this, it remains to show that the bottom map, ψ, is an isomorphism of $A[x]$-modules. It is clear that ψ is injective, and surjectivity follows from the equation $my^{-n} = xmy^{-(n-1)} - \psi(my^{-n})$ by induction on n.

To finish the proof, we construct φ. Note that the functor $S \mapsto A_\square[S]$ is monoidal, which means $A_\square[S] \overset{\mathrm{L}}{\otimes}_{A_\square} A_\square[T] \cong A_\square[S \times T]$ for any profinite sets S, T. As any A^I is a direct summand of $A_\square[S]$ for some profinite set S, we deduce a natural isomorphism

$$A^I \underset{A_\square}{\overset{\mathrm{L}}{\otimes}} A^J \cong A^{I\times J}, \quad \text{for all sets } I, J. \tag{1.13}$$

Let S be a profinite set. We compute

$$\begin{aligned}
\operatorname{Hom}_A\big(A[S], M^{(\mathbb{N})}\big) &\cong \operatorname{Hom}_A\big(A_\square[S], M^{(\mathbb{N})}\big) && (M^{(\mathbb{N})} \text{ is } A_\square\text{-analytic})\\
&\cong \bigoplus_{\mathbb{N}} \operatorname{Hom}_A\big(A_\square[S], M\big) && (A_\square[S] \text{ is compact in } \mathcal{D}_\square(A))\\
&\cong \operatorname{Hom}_A\big(A_\square[S]^{\mathbb{N}}, M\big)\\
&\cong \operatorname{Hom}_A\big(A_\square[S] \overset{\mathrm{L}}{\underset{A_\square}{\otimes}} A^{\mathbb{N}}, M\big) && (\text{by } (1.13))\\
&\cong \operatorname{Hom}_A\big(A[S] \overset{\mathrm{L}}{\underset{A}{\otimes}} A^{\mathbb{N}}, M\big) && (M \text{ is } A_\square\text{-analytic})\\
&\cong \operatorname{Hom}_A\big(A[S], \mathrm{R}\underline{\operatorname{Hom}}_A(A^{\mathbb{N}}, M)\big),
\end{aligned}$$

where the third isomorphism (which is given by $\sum_n f_n \mapsto \sum_n f_n \circ \mathrm{pr}_n$) follows from the fact that the functor $M \mapsto \underline{M}$ from topological A-modules into condensed A-modules is fully faithful, $A_\square[S]^{\mathbb{N}}$ carries the product topology, and M is discrete. These isomorphisms are induced by the map

$$\varphi'\colon M^{(\mathbb{N})} \longrightarrow \mathrm{R}\underline{\operatorname{Hom}}_A\big(A^{\mathbb{N}}, M\big), \tag{1.14}$$

which arises from the pairing $A^{\mathbb{N}} \otimes_A M^{(\mathbb{N})} \to M$ given by $(a_n)_n \otimes \sum_n m_n \mapsto \sum_n a_n m_n$. As the $A[S]$ generate $\mathcal{D}(A)$, we deduce that φ' is an isomorphism. Under the obvious identifications $M^{(\mathbb{N})} \cong M[y^{-1}]/M$ and $A^{\mathbb{N}} \cong A[\![y]\!]$ we obtain an isomorphism

$$\varphi\colon M[y^{-1}]/M \xrightarrow{\cong} \mathrm{R}\underline{\operatorname{Hom}}_A\big(A[\![y]\!], M\big), \tag{1.15}$$

arising from the pairing

$$\langle - \otimes - \rangle\colon A[\![y]\!] \otimes_A M[y^{-1}]/M \xrightarrow{\text{act}} M[y^{-1}]/M \xrightarrow{\text{res}} M,$$

where the *residue map* res is given by $\sum_{i\geq 0} m_{-i} y^{-i-1} \mapsto m_0$. To verify the commutativity of the bottom diagram above, it suffices to observe

$$x \cdot \Big\langle y \cdot \sum_{n\geq 0} a_n y^n \otimes \sum_{i\geq 0} m_{-i} y^{-i-1} \Big\rangle = \Big\langle \sum_{n\geq 0} a_n y^n \otimes \sum_{i\geq 0} x y m_{-i} y^{-i-1} \Big\rangle \qquad \text{in } M.$$

This finishes the proof. □

Proof of Theorem 1.6. Let A be a ring of finite type over $\mathbb{Z}$ and pick a surjection $\mathbb{Z}[x_1, \dots, x_n] \twoheadrightarrow A$. By Lemma 1.7.(a) we have $A_\square = (A, \mathbb{Z}[x_1, \dots, x_n])_\square$. Hence, by Lemma 1.7.(b) it suffices to prove that $\mathbb{Z}[x_1, \dots, x_n]_\square$ is analytic. By induction on n, we are reduced to showing the following statement:

Suppose that A is a finite-type $\mathbb{Z}$-algebra such that $A_\square$ is analytic; then $A[x]_\square$ is analytic.

Pick a complex $C = [\cdots \to C_n \to \cdots \to C_0 \to 0]$ of $A[x]$-modules, where each C_n is of the form $\bigoplus_I A[x]^J$, and fix a profinite set S. Consider the commutative diagram

$$\begin{tikzcd}
\mathrm{R}\underline{\operatorname{Hom}}_{A[x]}\big(A[x]_\square[S], C\big) \arrow[r] \arrow[dd] & \mathrm{R}\underline{\operatorname{Hom}}_{A[x]}\big(A[x][S], C\big) \arrow[d, "\cong"] \\
& \mathrm{R}\underline{\operatorname{Hom}}_{A}\big(A[S], C\big) \arrow[d, "\cong"] \\
\mathrm{R}\underline{\operatorname{Hom}}_{A[x]}\big(A[x] \otimes_A A_\square[S], C\big) \arrow[r, "\cong"'] & \mathrm{R}\underline{\operatorname{Hom}}_{A}\big(A_\square[S], C\big),
\end{tikzcd}$$

where the lower right vertical map is an isomorphism because C is $A_\square$-analytic. We have to show that the top map is an isomorphism. Consider the cokernel Q of the map $A[x] \otimes_A A_\square[S] \to A[x]_\square[S]$. We then have a fiber sequence $\mathrm{R}\underline{\operatorname{Hom}}_{A[x]}(Q, C) \to \mathrm{R}\underline{\operatorname{Hom}}_{A[x]}(A[x]_\square[S], C) \to \mathrm{R}\underline{\operatorname{Hom}}_{A[x]}(A[x] \otimes_A A_\square[S], C)$. We claim that

$$\mathrm{R}\underline{\operatorname{Hom}}_{A[x]}(Q, C) = 0. \tag{1.16}$$

Once this is proven, it follows that the left vertical map above is an isomorphism. But then also the top map is an isomorphism, which finishes the proof.

By Lemma 1.11.(a), we know that Q is an $A[x]_\infty$-module. Hence, we have $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(Q,C) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A[x]_\infty}\big(Q, \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C)\big)$, and it remains to prove $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C) = 0$. Write $C = \varinjlim_n \sigma^{\geq -n} C$ as a filtered colimit of its brutal truncations. Since $A[x]_\infty$ is compact in $\mathcal{D}_\square(A[x], A)$, by Lemma 1.11.(b), and since the full subcategory of compact objects in $\mathcal{D}_\square(A[x], A)$ is closed under tensor products by Remark 1.10, it follows formally that the natural map

$$\varinjlim_n \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}\big(A[x]_\infty, \sigma^{\geq -n} C\big) \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}\big(A[x]_\infty, C\big)$$

is an isomorphism. We may thus assume that C is bounded. Moreover, using that C is the cofiber of the map $\sigma^{\leq a} C[-1] \to \sigma^{>a} C$, we inductively reduce to the case $C = \bigoplus_I A[x]^J$ and then further to $C = A[x]$. But now, Lemma 1.11.(c) shows $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, A[x]) = 0$. We deduce (1.16), finishing the proof. $\square$

1.17. **Exercise.** Let A be a finitely generated $\mathbb{Z}$-algebra and $M \in \mathcal{D}_\square(A[x], A)$. Show:

(a) $M \otimes_{(A[x],A)_\square} A[x]_\square = 0 \iff M$ is an $A[x]_\infty$-module.
(b) $M \otimes_{(A[x],A)_\square} (A[x, x^{-1}], A[x^{-1}])_\square = 0 \iff M$ is an $A[\![x]\!]$-module.

Proof. We show (a). The "if"-direction follows from

$$\mathrm{Hom}_{A[x]}\big(M \otimes_{(A[x],A)_\square} A[x]_\square, C\big) \cong \mathrm{Hom}_{A[x]_\infty}\big(M, \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C)\big) = 0.$$

The last equation follows by writing $C = \varprojlim_n \sigma^{\leq n} C$, and then (1.16) implies $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, C) \cong \varprojlim_n \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[x]_\infty, \sigma^{\leq n} C) = 0$, because every $\sigma^{\leq n} C$ is quasi-isomorphic to some complex $[\cdots \to D_i \to \cdots \to D_n \to 0]$, where each D_i is of the form $\bigoplus_I A[x]^J$. For the "only if"-direction, it suffices to show that the cofiber of the unit map $M \to M \otimes_{(A[x],A)_\square} A[x]_\square$ is naturally an $A[x]_\infty$-module. But since both sides commute with colimits, we may reduce to the case $M = A[x] \otimes_A A[S]$, for some profinite set S, in which case this follows from Lemma 1.11.(a).

We now prove (b), which is similar to (a). First note that we have a short exact sequence

$$A[x] \otimes_A A[\![y]\!] \xrightarrow{x-y} A[x] \otimes_A A[\![y]\!] \longrightarrow A[\![x]\!] \longrightarrow 0, \tag{1.18}$$

which shows that $A[\![x]\!]$ is compact in $\mathcal{D}_\square(A[x], A)$. For the "if"-direction we compute, for any $C \in \mathcal{D}_\square(A[x, x^{-1}], A[x^{-1}])$,

$$\mathrm{Hom}_{A[x,x^{-1}]}\big(M \otimes_{(A[x],A)_\square} (A[x, x^{-1}], A[x^{-1}])_\square, C\big) \cong \mathrm{Hom}_{A[\![x]\!]}\big(M, \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[\![x]\!], C)\big) = 0.$$

For the last equation, we can reduce as before the the case $C = A[x, x^{-1}]$, and then the proof of Lemma 1.11.(c), but using the resolution (1.18), shows $\mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(A[\![x]\!], A[x, x^{-1}]) = 0$. For the "only if"-direction it suffices to show that the cofiber of $M \to M \otimes_{(A[x],A)_\square} (A[x, x^{-1}], A[x^{-1}])_\square$ is naturally an $A[\![x]\!]$-module. Again, we reduce to the case $M = A[x] \otimes_A A[S]$, for some profinite set S. Writing $A[S] \cong A^I$, we need to see that the cokernel of $A[x] \otimes_A A^I \to A[x, x^{-1}] \otimes_{A[x^{-1}]} A[x^{-1}]^I$, say Q, is an $A[\![x]\!]$-module. But since we have a pushout diagram

$$\begin{tikzcd} A[x] \otimes_A A^I \arrow[r] \arrow[d] & A[x, x^{-1}] \otimes_{A[x^{-1}]} A[x^{-1}]^I \arrow[d] \\ A[\![x]\!]^I \arrow[r] \arrow[ur, phantom, "\ulcorner"] & A((x))^I, \end{tikzcd}$$

this follows from the fact that Q is canonically isomorphic to the cokernel of the bottom map. $\square$

For later reference, we discuss the following lemma.

1.19. **Lemma.** *Let A be a finitely generated $\mathbb{Z}$-algebra. There is a natural equivalence*

$$- \otimes_{A_\square} A[x]_\square \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[x]_\infty / A[x], -\big)$$

of functors $\mathcal{D}_\square(A) \to \mathcal{D}_\square(A[x])$. In particular, $- \otimes_{A_\square} A[x]_\square$ is t-exact and preserves all limits.

Proof. The proof is very similar to the proof of Lemma 1.11.(c). Write $C \coloneqq A[x]_\infty/A[x]$ to shorten the notation. Note that $\mathrm{R}\underline{\mathrm{Hom}}_A(C,-) \cong \mathrm{R}\underline{\mathrm{Hom}}_{A[x]}(C, \mathrm{R}\underline{\mathrm{Hom}}_A(A[x],-))$ in $\mathcal{D}(A)$, so that the former naturally can be viewed as an object of $\mathcal{D}(A[x])$.

The A-linear map $M \otimes_A A[x]_\infty/A[x] \to M$, given by $m \otimes \sum_{i\geq 0} f_i x^{-i-1} \mapsto f_0 m$ induces an $A[x]$-linear map

$$M \otimes_A A[x] \longrightarrow \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[x]_\infty/A[x], M\big). \tag{1.20}$$

We have to show that the right-hand side lies in $\mathcal{D}_\square(A[x])$ whenever $M \in \mathcal{D}_\square(A)$, and in this case the map becomes an isomorphism after analytification.

For $M = A$ the map (1.20) is an isomorphism, which follows from (1.14) after the identifications $A[x] \cong A^{(\mathbb{N})}$ and $A[x]_\infty/A[x] \cong A^{\mathbb{N}}$. Since the latter is compact in $\mathcal{D}_\square(A)$, it follows that $\mathrm{R}\underline{\mathrm{Hom}}_A(A[x]_\infty/A,-)$ commutes with colimits. As $\mathcal{D}_\square(A)$ is generated by A under products and colimits, it follows that $\mathrm{R}\underline{\mathrm{Hom}}_A(A[x]_\infty/A[x], M)$ is $A[x]_\square$-analytic for all $M \in \mathcal{D}_\square(A)$. We obtain a morphism

$$M \otimes_{A_\square} A[x]_\square \longrightarrow \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[x]_\infty/A[x], M\big) \tag{1.21}$$

of $A[x]_\square$-analytic modules. Now, both sides commute with colimits in M. In order to show that (1.21) is an isomorphism, we may thus reduce to $M = A^I$. Now, we have a commutative diagram

$$\begin{tikzcd}
A^I \otimes_{A_\square} A[x]_\square \arrow[r] \arrow[d] & \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[x]_\infty/A[x], A^I\big) \arrow[d, "\cong"] \\
\prod_I A[x] \arrow[r, "\cong"'] & \prod_I \mathrm{R}\underline{\mathrm{Hom}}_A\big(A[x]_\infty/A[x], A\big),
\end{tikzcd}$$

where the bottom map is the isomorphism (1.20) for $M = A$ in each component. The left vertical map is an isomorphism: we may enlarge I and hence assume that $A^I \cong A_\square[S]$, for some profinite set S; the map then identifies with the isomorphism $A_\square[S] \otimes_{A_\square} A[x]_\square \xrightarrow{\cong} A[x]_\square[S]$. Consequently, the top map in the diagram, hence also (1.21), is an isomorphism as well. $\square$

2. Animated Analytic Rings

2.1. We briefly recall the notion of an *animated condensed ring*. We write An for the ∞-category of *anima*.

Let $\mathcal{C}$ be the category of condensed rings of the form $\mathbb{Z}[\mathbb{N}[S]]$, where $\mathbb{N}[S]$ is the free abelian monoid on the extremally disconnected set S; thus $\mathbb{Z}[\mathbb{N}[S]]$ is the free condensed ring on S. The retracts of objects in $\mathcal{C}$ are precisely the compact projective objects in $\mathrm{Cond}(\mathsf{Ring})$.

The ∞-category $\mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring})) \coloneqq \mathrm{Fun}^{\Pi}(\mathcal{C}^{\mathrm{op}}, \mathsf{An})$ of functors which commute with finite products is called the *animation* of $\mathrm{Cond}(\mathsf{Ring})$.

Every condensed ring A determines a functor $\mathcal{C}^{\mathrm{op}} \to \mathsf{Set} \subseteq \mathsf{An}$ via $R \mapsto \mathrm{Hom}_{\mathcal{C}}(R, A)$. This determines a fully faithful embedding $\mathrm{Cond}(\mathsf{Ring}) \hookrightarrow \mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring}))$. Animated condensed rings satisfy the following universal property: let $\mathcal{D}$ be an ∞-category which admits all sifted colimits. The restriction map

$$\mathrm{Fun}^{\mathrm{sifted}}\big(\mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring})), \mathcal{D}\big) \longrightarrow \mathrm{Fun}(\mathrm{Cond}(\mathsf{Ring}), \mathcal{D})$$

is an equivalence of functor ∞-categories, where the left-hand side denotes the full subcategory of those functors $\mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring})) \to \mathcal{D}$ which preserve sifted colimits. In other words: $\mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring}))$ is obtained by freely adjoining all sifted colimits to $\mathcal{C}$.

We now recall the notion of analytic rings from the last talk.

2.2. **Definition.** An *analytic ring* is a pair $\mathcal{A} = (\underline{\mathcal{A}}, \mathcal{M})$, where $\underline{\mathcal{A}}$ is an animated condensed ring and $\mathcal{M}$ is a functor

$$\mathcal{M} \colon (\mathsf{ExtrDisc}) \longrightarrow \mathcal{D}^{\leq 0}(\underline{\mathcal{A}}),$$

together with a natural transformation $\underline{\mathcal{A}}[S] \to \mathcal{A}[S] \coloneqq \mathcal{M}(S)$, such that the following properties are satisfied:

(a) $\mathcal{A}[S \sqcup T] = \mathcal{A}[S] \oplus \mathcal{A}[T]$ and the map $\underline{\mathcal{A}} \to \mathcal{A}[*]$ is an isomorphism of $\underline{\mathcal{A}}$-modules.

(b) If $M \in \mathcal{D}(\underline{\mathcal{A}})$ is a sifted colimit of copies of $\mathcal{A}[S]$, for varying extremally disconnected sets S, then for all $S \in (\mathsf{ExtrDisc})$ the natural map

$$\mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}\big(\mathcal{A}[S], M\big) \xrightarrow{\cong} \mathrm{R}\underline{\mathrm{Hom}}_{\underline{\mathcal{A}}}\big(\underline{\mathcal{A}}[S], M\big)$$

is an isomorphism.

(c) For every prime p, the forgetful functor along the Frobenius $\underline{\mathcal{A}} \to \underline{\mathcal{A}}/p$ maps every $(\mathcal{A}/p)[S] := \mathcal{A}[S]/p$ to an $\underline{\mathcal{A}}$-module M satisfying the property in (b).

The modules $M \in \mathcal{D}(\underline{\mathcal{A}})$ satisfying the property in (b) are called $\mathcal{A}$-*analytic*. We denote the full subcategory of $\mathcal{A}$-analytic modules by $\mathcal{D}(\mathcal{A})$.

A morphism $\mathcal{A} \to \mathcal{B}$ of analytic rings is a morphism $\underline{\mathcal{A}} \to \underline{\mathcal{B}}$ of underlying animated condensed rings such that the image of $\mathcal{D}(\mathcal{B})$ maps into $\mathcal{D}(\mathcal{A})$ under the forgetful functor.

2.3. **Remark.** (a) We are mostly interested in discrete animated rings. These are precisely the sifted colimits of the polynomial rings $\mathbb{Z}[x_1, \dots, x_n]$. The full subcategory of $\mathrm{Ani}(\mathrm{Cond}(\mathsf{Ring}))$ consisting of discrete animated rings is denoted $\mathrm{Ani}(\mathsf{Ring})$.

(b) Analytic rings assemble into an ∞-category AnRing, which admits all limits and colimits.

(c) The analytic ring structure $\mathcal{A}$ is uniquely determined by the full subcategory $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$, as $\mathcal{A}[S]$ is recovered as the image of $\underline{\mathcal{A}}[S]$ under the left adjoint of the inclusion.

(d) The t-structure on $\mathcal{D}(\underline{\mathcal{A}})$ restricts to a t-structure on $\mathcal{D}(\mathcal{A})$ whose heart $\mathcal{D}(\mathcal{A})^{\heartsuit}$ is generated under colimits (in $\mathrm{Mod}(\pi_0\underline{\mathcal{A}})$) by the $\pi_0\mathcal{A}[S]$, for varying extremally disconnected S, cf. [Man22, Proposition 2.3.8]. This means that $M \in \mathcal{D}(\underline{\mathcal{A}})$ is $\mathcal{A}$-analytic if and only if $H^n(M) \in \mathcal{D}(\mathcal{A})^{\heartsuit}$ for all $n \in \mathbb{Z}$.

Upshot: The full subcategory $\mathcal{D}(\mathcal{A}) \subseteq \mathcal{D}(\underline{\mathcal{A}})$ is completely determined by a full subcategory of $\mathrm{Mod}(\pi_0\underline{\mathcal{A}})$.

2.4. **Definition.** (a) Let A be a discrete animated ring. We put $A_\square := \varinjlim_i A_{i,\square}$, where $A = \varinjlim_i A_i$ is any representation of A as a sifted colimit of classical rings that are of finite type over $\mathbb{Z}$.[1] Concretely, for any extremally disconnected set S, we have $A_\square[S] = \varinjlim_i A_i^J$ for a suitable set J.

(b) Let $A \to B$ be a morphism of discrete animated rings. We denote $(B, A)_\square$ the analytic ring structure on B given by $(B, A)_\square[S] := B \otimes_A A_\square[S]$.

(c) We denote $\mathcal{D}_\square(A) := \mathcal{D}(A_\square)$ and $\mathcal{D}_\square(B, A) := \mathcal{D}((B, A)_\square)$. Recall that $\mathcal{D}_\square(B, A)$ is the full subcategory of $\mathcal{D}(B)$ consisting of condensed B-modules which are $A_\square$-analytic.

2.5. **Lemma.** *Let $A \to B$ be a morphism of classical rings and denote $\widetilde{A} \subseteq B$ the integral closure of the image of A in B. Then $(B, A)_\square = (B, \widetilde{A})_\square$.*

Proof. Let S be a profinite set. We have to show that the canonical map

$$B \otimes_A A_\square[S] \longrightarrow B \otimes_{\widetilde{A}} \widetilde{A}_\square[S]$$

is an isomorphism. Clearly, we may assume $B = \widetilde{A}$; in other words, we reduce to the situation where $A \to B$ is integral and have to show that the map $B \otimes_A A_\square[S] \to B_\square[S]$ is an isomorphism.

We show that $A \to B$ can be written as a direct limit of finite maps between finitely generated algebras. Let I be the set of finite subsets of B, ordered by inclusion. Then $B = \varinjlim_{i \in I} A[i]$. Since $A \to B[i]$ is integral, we find a finite subset $\Lambda_i \subseteq A$ such that each element of i is integral over $A_{i,0} := \mathbb{Z}[\Lambda_i]$ and such that $\Lambda_{i_1} \subseteq \Lambda_{i_2}$ whenever $i_1 \subseteq i_2$. Let J be the set of finite subsets of A, ordered by inclusion. The set $I \times J$ is directed with partial order given by $(i_1, j_1) \leq (i_2, j_2)$ if $i_1 \subseteq i_2$ and $j_1 \subseteq j_2$. Put $A_{i,j} := A_{i,0}[j]$ and $B_{i,j} := A_{i,0}[i \cup j]$, for any $(i, j) \in I \times J$. Then $A_{i,j} \to B_{i,j}$ is a finite map between finitely generated algebras, and $A \to B$ is the direct limit of the maps $A_{i,j} \to B_{i,j}$.

[1] More rigorously, $\mathrm{Ani}(\mathsf{Ring}) \to \mathsf{AnRing}$, $A \mapsto A_\square$, is the unique extension of $\{$polynomial rings$\} \to \mathsf{AnRing}$, $\mathbb{Z}[x_1, \dots, x_n] \mapsto \mathbb{Z}[x_1, \dots, x_n]_\square$, to a functor that preserves sifted colimits. With this description it is clear that $A_\square$ is independent of the chosen presentation of A as a sifted colimit of finitely generated algebras.

For any extremally disconnected set S we compute

$$\begin{aligned} B_\square[S] &\cong \varinjlim_{(i,j)} (B_{i,j})_\square[S] \cong \varinjlim_{(i,j)} \big(B_{i,j} \otimes_{A_{i,j}} (A_{i,j})_\square[S]\big) \\ &\cong \varinjlim_{(i,j)} B_{i,j} \otimes_{\varinjlim A_{i,j}} \varinjlim_{(i,j)} (A_{i,j})_\square[S] \cong B \otimes_A A_\square[S], \end{aligned}$$

where the second isomorphism uses Lemma 1.7.(a) and the third isomorphism follows from the fact that $I \times J$ is directed. □

2.6. **Definition.** (a) A *discrete Huber pair* is a pair (A, A^+) consisting of a discrete animated ring A and an integrally closed subring $A^+ \subseteq \pi_0 A$.

A morphism $(A, A^+) \to (B, B^+)$ of discrete Huber pairs is a morphism $A \to B$ of animated rings such that A^+ maps to B^+ under the induced map $\pi_0 A \to \pi_0 B$.

(b) Let (A, A^+) be a discrete Huber pair. We define an analytic ring structure $(A, A^+)_\square$ on A as follows: if A is static, we put $(A, A^+)_\square[S] := A \otimes_{A^+} A^+_\square[S]$ for any extremally disconnected set S. For general A, we let $(A, A^+)_\square$ be the analytic ring structure induced from $(\pi_0 A, A^+)_\square$, cf. [Man22, Proposition 2.3.14].

(c) We write $\mathcal{D}_\square(A, A^+) = \mathcal{D}((A, A^+)_\square)$. An $(A, A^+)_\square$-analytic module is an object of $\mathcal{D}_\square(A, A^+)$.

2.7. **Notation.** Given discrete animated rings A and A^+ and a map $A^+ \to \pi_0 A$, we write

$$(A, A^+)_\square := (A, \widetilde{A^+})_\square,$$

where $\widetilde{A^+} \subseteq \pi_0 A$ is the integral closure of the image of the map $\pi_0 A^+ \to \pi_0 A$.

2.8. **Remark.** For a general discrete Huber pair (A, A^+), the analytic ring structure $(A, A^+)_\square$ is difficult to describe explicitly. However, the category $\mathcal{D}_\square(A, A^+)$ admits an easy concrete description: It consists of those modules $M \in \mathcal{D}(A)$ such that $H^n(M) \in \mathcal{D}_\square(\pi_0 A, A^+)$ for all $n \in \mathbb{Z}$.

2.9. **Remark.** The ∞-category AnRing of analytic rings is cocomplete, and while sifted colimits are easy to compute, the description of pushouts is quite subtle, cf. [Man22, Proposition 2.3.15]. Given morphisms $\mathcal{B} \leftarrow \mathcal{A} \to \mathcal{C}$ in AnRing, the pushout $\mathcal{E} = \mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}$ is the "completion" of the uncompleted analytic ring structure on $\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{C}}$ such that $\mathcal{D}(\mathcal{E}) \subseteq \mathcal{D}(\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{C}})$ is given by all modules $M \in \mathcal{D}(\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{C}})$ which map to $\mathcal{D}(\mathcal{B})$ and $\mathcal{D}(\mathcal{C})$ under the forgetful functors. If S is an extremally disconnected set, then $\mathcal{E}[S]$ is given as the colimit of the repeated application of $(- \otimes_{\underline{\mathcal{B}}} \mathcal{B}) \otimes_{\underline{\mathcal{C}}} \mathcal{C}$ to $(\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{C}})[S]$.

A particularly nice class of morphisms in AnRing, along which the base change can be easily described, are the so-called *steady* ones: a morphism $f\colon \mathcal{A} \to \mathcal{B}$ in AnRing is called *steady* if for all morphisms $g\colon \mathcal{A} \to \mathcal{C}$ of analytic rings and all extremally disconnected sets S the canonical map

$$\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}[S] \xrightarrow{\cong} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C})[S]$$

is an isomorphism in $\mathcal{D}(\underline{\mathcal{B}} \otimes_{\underline{\mathcal{A}}} \underline{\mathcal{C}})$. Equivalently, for all $g\colon \mathcal{A} \to \mathcal{C}$ as above, and all $M \in \mathcal{D}(\mathcal{C})$ the canonical map $\mathcal{B} \otimes_{\mathcal{A}} M \to (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{C}) \otimes_{\mathcal{C}} M$ is an isomorphism in $\mathcal{D}(\underline{\mathcal{B}})$.

2.10. **Proposition.** *(a) The assignment $(A, A^+) \mapsto (A, A^+)_\square$ defines a fully faithful functor from the ∞-category of discrete Huber pairs to AnRing, which preserves all non-empty colimits. In particular, for any morphisms $(B, B^+) \leftarrow (A, A^+) \to (C, C^+)$ of discrete Huber pairs, we have*

$$(B, B^+)_\square \otimes_{(A,A^+)_\square} (C, C^+)_\square = (B \otimes_A C, B^+ \otimes_{A^+} C^+)_\square.$$

(b) Every morphism $f\colon (A, A^+)_\square \to (B, B^+)_\square$ is steady*, i.e., for all pushout diagrams*

$$\begin{tikzcd} (A, A^+)_\square \arrow[r, "f"] \arrow[d, "g"'] & (B, B^+)_\square \arrow[d, "g'"] \\ \mathcal{C} \arrow[r, "f'"'] \arrow[ur, phantom, "\ulcorner", near end] & \mathcal{C}' \end{tikzcd}$$

of analytic rings and all $M \in \mathcal{D}(\mathcal{C})$ the natural map

$$M \otimes_{(A,A^+)_\square} (B, B^+)_\square \xrightarrow{\cong} M \otimes_{\mathcal{C}} \mathcal{C}'$$

is an isomorphism in $\mathcal{D}_{\square}(B, B^+)$.

Proof. For (a), see [Man22, Proposition 2.9.6]. We only remark that the identity on tensor products in (a) is reduced to the case $A = A^+$, $B = B^+ = A[x]$ and $C = C^+$ and where A, B, C are of finite type. For any extremally disconnected set S the object $(A[x]_\square \otimes_{A_\square} C_\square)[S] \in \mathcal{D}(C[x])$ is given as the colimit of the repeated applications of $(- \otimes_C C_\square) \otimes_{A[x]} A[x]_\square$ to $A[x] \otimes_A C[S]$. We have $((A[x] \otimes_A C[S]) \otimes_C C_\square) \otimes_{A[x]} A[x]_\square = (A[x] \otimes_A C_\square[S]) \otimes_{A[x]} A[x]_\square = A[x]_\square \otimes_{A_\square} C_\square[S]$, and then (1.5) and Lemma 1.19 show that the natural map

$$A[x]_\square \otimes_{A_\square} C_\square[S] \xrightarrow{\cong} C[x]_\square[S]$$

is an isomorphism. We deduce $(A[x]_\square \otimes_{A_\square} C_\square)[S] \cong C[x]_\square[S]$, which proves the claim.

For (b) we refer to [Man22, Proposition 2.9.7]. $\square$

3. Gluing discrete Huber pairs

3.1. The goal in this section is to concoct a geometric theory of *discrete adic spaces* from the analytic rings $(A, A^+)_\square$. This is done by specifying "standard immersions" $(A, A^+)_\square \to (B, B^+)_\square$ along which we can glue these analytic rings.[2] This gluing procedure is completely formal, and so we explain the general scheme before applying it to the analytic rings $(A, A^+)_\square$.

3.2. **Definition.** A morphism $f\colon \mathcal{A} \to \mathcal{B}$ of analytic rings is called a *localization* if the forgetful functor $f_*\colon \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ is fully faithful.

One should think of (steady) localizations as "standard immersions". These have nice closure properties as the next result shows.

3.3. **Proposition.** *(a) Localizations of analytic rings are stable under composition and base change. Steady localizations are stable under all colimits.*

(b) Let $f\colon \mathcal{A} \to \mathcal{B}$ be a morphism of analytic rings. If f is a localization, then the induced map $\mathcal{B} \xrightarrow{\cong} \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ is an isomorphism. Conversely, if f is steady and $\mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ is an isomorphism then f is a localization.

Proof. We only show (b); for (a) we refer to [Man22, Proposition 2.4.2]. So let $f\colon \mathcal{A} \to \mathcal{B}$ be a localization. Let $\mathcal{B}'$ be the uncompleted analytic ring given by $\underline{\mathcal{B}}' = \underline{\mathcal{B}}$ and $\mathcal{B}'[S] = (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B})[S]$. Since f_* is fully faithful, the counit $\mathcal{B}[S] \otimes_{\mathcal{A}} \mathcal{B} = f^* f_*(\mathcal{B}[S]) \to \mathcal{B}[S]$ is an isomorphism, hence so is the unit $\mathcal{B}[S] \xrightarrow{\cong} \mathcal{B}[S] \otimes_{\mathcal{A}} \mathcal{B}$. Therefore, $\mathcal{B}[S] \otimes_{\mathcal{A}} \mathcal{B}$ maps to $\mathcal{D}(\mathcal{B})$ under both projections, and we deduce $\mathcal{B}'[S] \cong \mathcal{B}[S] \otimes_{\mathcal{A}} \mathcal{B} \cong \mathcal{B}[S]$. But then $\mathcal{B}'$ is already completed so that $\mathcal{B} \xrightarrow{\cong} \mathcal{B}' \cong \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$.

Conversely, if f is steady and $\mathcal{B} \to \mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}$ is an isomorphism, then for all $M \in \mathcal{D}(\mathcal{B})$ we have $M \otimes_{\mathcal{A}} \mathcal{B} \cong M \otimes_{\mathcal{B}} (\mathcal{B} \otimes_{\mathcal{A}} \mathcal{B}) \cong M \otimes_{\mathcal{B}} \mathcal{B} \cong M$, where the first isomorphism uses the steadiness of f. Hence, the counit $f^* f_* \to \mathrm{id}_{\mathcal{D}(\mathcal{B})}$ is an isomorphism, which is equivalent to $f_*\colon \mathcal{D}(\mathcal{B}) \to \mathcal{D}(\mathcal{A})$ being fully faithful. $\square$

3.4. **Definition.** A *geometry blueprint* is a pair $G = (R_G, L_G)$, where $R_G \subseteq \mathsf{AnRing}$ is a full subcategory and L_G is a class of morphisms in R_G, such that the following properties are satisfied:

(a) We have $\{\text{equivalences}\} \subseteq L_G \subseteq \{\text{steady localizations}\}$.
(b) L_G is stable under composition and arbitrary base change in R_G.
(c) R_G is stable under pushouts and finite products in AnRing. In particular, R_G contains the terminal object 0.

The objects of R_G are calld *G-analytic rings* and the morphisms in L_G are called *G-localizations*.

We fix a geometry blueprint $G = (R_G, L_G)$.

3.5. **Definition.** (a) The objects of R_G^{op} are denoted $\mathrm{AnSpec}_G\, \mathcal{A}$, for $\mathcal{A} \in R_G$.

(b) The *G-analytic site* on R_G^{op} is the Grothendieck site whose coverings are generated by finite families of G-localizations $\{\mathrm{AnSpec}_G\, \mathcal{A}_i \to \mathrm{AnSpec}_G\, \mathcal{A}\}_i$ such that the pullback $\mathcal{D}(\mathcal{A}) \to \prod_i \mathcal{D}(\mathcal{A}_i)$ is conservative.

[2] These standard immersions, called G-localizations below, should NOT be thought of as open immersions. In fact, we should also allow adic spaces to be glued along closed immersions.

(c) We denote $\mathrm{Shv}(R_G^{\mathrm{op}})$ the ∞-category of sheaves (in anima) on the site R_G^{op}. More concretely, $\mathcal{F}\colon R_G \to \mathsf{An}$ is a *sheaf* if:
- $\mathcal{F}$ preserves finite products;
- $\mathcal{F}$ satisfies Čech descent: for every $X = \mathrm{AnSpec}_G \mathcal{A} \in R_G^{\mathrm{op}}$ and every covering $\{\mathrm{AnSpec}_G \mathcal{A}_i \to X\}_i$, consider the Čech nerve $U_\bullet\colon \Delta^{\mathrm{op}} \to R_G^{\mathrm{op}}$ given by $U_0 = \bigsqcup_i \mathrm{AnSpec}_G \mathcal{A}_i$ and $U_n \coloneqq U_0 \times_X \cdots \times_X U_0$ (which consists of $n+1$ copies of U_0); then the natural map

$$\begin{aligned}\mathcal{F}(X) &\xrightarrow{\cong} \varprojlim_{n\in\Delta} \mathcal{F}(U_n)\\ &\cong \varprojlim \Big(\mathcal{F}(U_0) \rightrightarrows \mathcal{F}(U_1) \rightrightarrows \mathcal{F}(U_2) \rightrightarrows \cdots \Big)\end{aligned}$$

is an isomorphism of anima.

3.6. **Example.** (a) The functor

$$\mathcal{D}\colon R_G \longrightarrow \mathrm{Cat}_\infty, \qquad \mathcal{A} \longmapsto \mathcal{D}(\mathcal{A})$$

is a sheaf valued in the ∞-category Cat_∞ of ∞-categories for the G-analytic site.

(b) For every $\mathcal{A} \in R_G$, the functor

$$\mathrm{AnSpec}_G \mathcal{A}\colon R_G \longrightarrow \mathsf{An}, \qquad \mathcal{B} \longmapsto \mathrm{Hom}_{R_G}(\mathcal{A}, \mathcal{B})$$

is a sheaf for the G-analytic site. Inparticular, we can view R_G^{op} as a full subcategory of $\mathrm{Shv}(R_G^{\mathrm{op}})$.

Proof. We prove (a). It is clear that the functor $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})$ commutes with finite products. Let $\{\mathrm{AnSpec}_G \mathcal{B}_i \to \mathrm{AnSpec}_G \mathcal{B}\}_{i\in I}$ be a G-analytic covering, where I is a finite set. For any non-empty subset $J \subseteq I$, we consider $f_J\colon \mathcal{B} \to \mathcal{B}_J \coloneqq \bigotimes_{j\in J} \mathcal{B}_j$, where the pushouts are taken over $\mathcal{B}$. Since each $\mathcal{B} \to \mathcal{B}_j$ is a steady localization, it follows inductively that $\mathcal{D}(\mathcal{B}_J) = \bigcap_{j\in J} \mathcal{D}(\mathcal{B}_j)$ as full subcategories of $\mathcal{D}(\mathcal{B})$. Unraveling the sheaf condition, we have to show that the natural map

$$F\colon \mathcal{D}(\mathcal{B}) \longrightarrow \varprojlim_{J\subseteq I} \mathcal{D}(\mathcal{B}_J), \qquad M \longmapsto \big(f_J^* M\big)_J$$

is an equivalence of ∞-categories, where the transition maps are given by the base change functors $f_{J,J'}^*\colon \mathcal{D}(\mathcal{B}_J) \to \mathcal{D}(\mathcal{B}_{J'})$ for $J \subseteq J'$. Note that the functor F admits a right adjoint sending $(N_J)_J \mapsto \varprojlim_J f_{J,*} N_J$, because

$$\begin{aligned}\mathrm{Hom}_{\varprojlim \mathcal{D}(\mathcal{B}_J)}\big(FM, (N_J)_J\big) &\cong \varprojlim_J \mathrm{Hom}_{\mathcal{D}(\mathcal{B}_J)}\big(f_J^* M, N_J\big)\\ &\cong \varprojlim_J \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}(M, f_{J,*}N_J) \cong \mathrm{Hom}_{\mathcal{D}(\mathcal{B})}\big(M, \varprojlim_J f_{J,*}N_J\big).\end{aligned}$$

Since $\mathcal{D}(\mathcal{B}) \to \prod_i \mathcal{D}(\mathcal{B}_i)$ is conservative, verifying that the unit $M \to \varprojlim_J f_{J,*} f_J^* M$ is an isomorphism can be checked after applying f_i^* for all $i \in I$. As the limit is finite, it commutes with the exact functor f_i^*. The unit becomes $f_i^* M \to \varprojlim_J f_i^* f_{J,*} f_J^* M \cong \varprojlim_{J\ni i} f_{J,*} f_J^* f_i^* M$, where the isomorphism uses the base change property of the steady map $\mathcal{B} \to \mathcal{B}_i$; this is an isomorphism, since $\{i\}$ is an initial element in $\{J \subseteq I \mid i \in J\}$. For the counit, we argue similarly. We have to show that for every $(N_J)_J \in \varprojlim_J \mathcal{D}(\mathcal{B}_J)$ and any $K \subseteq I$ the natural map $f_K^* \varprojlim_{J\subseteq I} f_{J,*} N_J \to N_K$, given as the adjoint of the natural projection, is an isomorphism. As $f_K^* N_J \cong N_{K\cup J}$, we have by a similar argument as above isomorphisms $f_K^* \varprojlim_J f_{J,*} N_J \cong \varprojlim_{J\supseteq K} f_{J,*} N_J \xrightarrow{\cong} N_K$, because K is initial in $\{J \subseteq I \mid K \subseteq J\}$. We conclude that $\mathcal{A} \mapsto \mathcal{D}(\mathcal{A})$ is a sheaf.

In order to prove (b), let $\mathcal{B}, \mathcal{B}_i, \mathcal{B}_J$ be as above. Given $\mathcal{A} \in R_G$, we have to show that the natural map

$$\mathrm{Hom}_{R_G}(\mathcal{A}, \mathcal{B}) \longrightarrow \varprojlim_{J\subseteq I} \mathrm{Hom}_{R_G}(\mathcal{A}, \mathcal{B}_J) \cong \mathrm{Hom}_{R_G}\big(\mathcal{A}, \varprojlim_J \mathcal{B}_J\big)$$

is an isomorphism. It suffices to show that $\mathcal{B} \to \varprojlim_J f_{J,*}\mathcal{B}_J$ is an isomorphism in AnRing. By (a), it follows that the map $\mathcal{B}[S] \to \varprojlim_J f_{J,*}\mathcal{B}_J[S]$ is an isomorphism in $\mathcal{D}(\mathcal{B})$, for all extremally disconnected sets S. Setting $S = *$ and using that the forgetful functor from algebras to modules is conservative, we deduce $\mathcal{B} \xrightarrow{\cong} \varprojlim_J \mathcal{B}_J$ as analytic rings. $\square$

3.7. **Definition.** Let $X \in \mathrm{Shv}(R_G^{\mathrm{op}})$ be a sheaf.

(a) X is called an *affine G-analytic space* if $X \cong \mathrm{AnSpec}_G \mathcal{A}$, for some analytic ring $\mathcal{A} \in R_G$.
(b) Assume that $X = \mathrm{AnSpec}_G \mathcal{A}$ is affine. A *G-analytic subspace* of X is a subsheaf $U \subseteq X$ which admits a cover[3] $\coprod_i U_i \twoheadrightarrow U$ by a small family of affine subsheaves $U_i \cong \mathrm{AnSpec}_G \mathcal{B}_i \subseteq U$, such that the induced maps $\mathcal{A} \to \mathcal{B}_i$ are G-localizations.
(c) A *G-analytic subspace* of X is a subsheaf $U \subseteq X$ such that for every map $Y \to X$ from an an affine G-analytic space Y, the pullback $U \times_X Y \to Y$ is a G-analytic subspace.
(d) X is called a *G-analytic space* if it can be covered by subsheaves which are affine G-analytic subspaces.

3.8. **Remark.** Since G-localizations are stable under all base changes, the definitions (b) and (c) are compatible.

3.9. **Definition.** An *analytic space* is a G_0-analytic space for the geometry blueprint $G_0 = (\mathsf{AnRing}, \{\text{steady localizations}\})$. We write $\mathsf{AnSpace}$ for the ∞-category of analytic spaces. A *steady subspace* of an analytic space is a G_0-analytic subspace.

We finally come to the definition of discrete adic spaces.

3.10. **Definition.** (a) Let $G_{\omega\text{-adic}}$ be the geometry blueprint whose $G_{\omega\text{-adic}}$-analytic rings are the analytic rings $(A, A^+)_\square$ for discrete Huber pairs (A, A^+) and whose $G_{\omega\text{-adic}}$-localizations are generated by the maps

$$(A, A^+)_\square \longrightarrow \big(A[1/g], A^+[f_1/g, \ldots, f_n/g]\big)_\square,$$

where $f_1, \ldots, f_n, g \in \pi_0 A$.

(b) A *discrete adic space* is a $G_{\omega\text{-adic}}$-analytic space. For every discrete Huber pair (A, A^+), we denote

$$\mathrm{Spa}(A, A^+) \coloneqq \mathrm{AnSpec}_{G_{\omega\text{-adic}}}(A, A^+)_\square.$$

An *immersion* $U \hookrightarrow X$ of discrete adic spaces is a map which exhibits U as a $G_{\omega\text{-adic}}$-analytic subspace of X.

(c) A discrete adic space is called *classical* if it can be covered by open subspaces of the form $\mathrm{Spa}(A, A^+)$ for static A.

3.11. **Remark.** The $G_{\omega\text{-adic}}$-localizations are generated under base change and composition by the maps

$$(\mathbb{Z}[x], \mathbb{Z})_\square \longrightarrow (\mathbb{Z}[x, x^{-1}], \mathbb{Z})_\square \qquad \text{and} \qquad (\mathbb{Z}[x], \mathbb{Z})_\square \longrightarrow \mathbb{Z}[x]_\square.$$

3.12. **Definition.** (a) We let G_{sch} denote the geometry blueprint whose G_{sch}-analytic rings are precisely the analytic rings of the form $A_\square$, for discrete animated rings A, and whose G_{sch}-localizations are generated by the map $\mathbb{Z}[x]_\square \to \mathbb{Z}[x, x^{-1}]_\square$.

(b) A *scheme* is a G_{sch}-analytic space. For every discrete animated ring we write

$$\mathrm{Spec}\, A \coloneqq \mathrm{AnSpec}_{G_{\mathrm{sch}}} A_\square.$$

An *open immersion* $U \hookrightarrow X$ of schemes is a map which exhibits U as a G_{sch}-analytic subspace of X.

(c) A scheme is called *classical* if it can be covered by open subspaces of the form $\mathrm{Spec}\, A$ where A is static.

4. Comparison with Huber's discrete adic spaces

4.1. At last, we shortly discuss how the analytic theory developed here compares with the classical theory of discrete adic spaces. We will not discuss the topology on our new adic spaces but only mention that, unlike in Huber's theory, the localization $(\mathbb{Z}[x], \mathbb{Z})_\square \to (\mathbb{Z}[x, x^{-1}], \mathbb{Z})_\square$ is *not* an open immersion in our new setting. This change is necessary to make the six functor formalism work.

[3] A morphism of sheaves $\mathcal{F} \twoheadrightarrow \mathcal{G}$ is a *cover* if it is an effective epimorphism, that is, $\mathcal{G} \cong \varinjlim_{n \in \Delta^{\mathrm{op}}} \mathcal{F}_n$, where $\mathcal{F}_\bullet$ denotes the Čech nerve of $\mathcal{F} \to \mathcal{G}$; this is the ∞-categorical analog of saying that $\mathcal{F} \times_{\mathcal{G}} \mathcal{F} \rightrightarrows \mathcal{F} \to \mathcal{G}$ is a (reflexive) coequalizer.

The classical theory is built via locally ringed spaces, where coverings are defined as topological coverings. In contrast, we have defined coverings in terms of conservative pullbacks. In order to reconcile these notions, we introduce the underlying set $|X|$ of a discrete adic space X as

$$|X| := \left\{ \mathrm{Spa}(K, K^+) \to X \;\middle|\; \begin{array}{l} K^+ \text{ is a valuation ring} \\ \text{with fraction field } K \end{array} \right\}_{/\sim}.$$ [4]

The inclusion of classical discrete adic spaces into discrete adic spaces admits a right adjoint, written $X \mapsto X_0$, which is completely determined by the assignment $\mathrm{Spa}(A, A^+) \mapsto \mathrm{Spa}(\pi_0 A, A^+)$. As any morphism $\mathcal{A} \to K$ factors through $\pi_0\mathcal{A}$, it is clear from the definition that $|X| \cong |X_0|$ depends only on the underlying classical discrete adic space.

Let us describe an important family of subsets which, in Huber's theory, would give a basis of a topology on $|X_0|$. For this, we may assume $X_0 = \mathrm{Spa}(A, A^+)$, where A is a classical ring. Consider the *rational domains*

$$U\Big(\frac{T}{g}\Big) := \mathrm{Spa}\big(A[1/g], A^+[t_1/g, \dots, t_n/g]\big),$$

where $T = \{t_1, \dots, t_n\} \subseteq \pi_0 A$ is a finite subset; note that we may replace T with $T \cup \{g\}$ without changing the rational domain. A point $x \in |X|$ is given by a map $\pi_0 A \to K$ sending $A^+ \to K^+$; composing with the valuation $K \to \Gamma \cup \{0\}$, we obtain a multiplicative valuation $|(\cdot)(x)| \colon \pi_0 A \to \Gamma \cup \{0\}$ such that $|f(x)| \le 1$ for all $f \in A^+$. Conversely, any such valuation corresponds to a point of $|X_0|$. We then have

$$x \in \left| U\Big(\frac{T}{g}\Big) \right| \iff |t_i(x)| \le |g(x)| \text{ for all } i = 1, \dots, n.$$

Our comparison result relies on the following "adic Nullstellensatz".

4.2. **Proposition.** *Let A be a discrete animated ring.*

(a) One has an injective map

$$\begin{array}{ccc} \left\{ \begin{array}{c} \textit{integrally closed} \\ \textit{subrings } A^+ \subseteq \pi_0 A \end{array} \right\} & \hookrightarrow & \big\{ \textit{subsets of } |\mathrm{Spa}(A, \mathbb{Z})| \big\} \\ A^+ & \longmapsto & |\mathrm{Spa}(A, A^+)|. \end{array}$$

(b) Let $\alpha \colon (A, A^+) \to (B, B^+)$ be a map of discrete Huber pairs such that the induced map $|\mathrm{Spa}(B, B^+)| \to |\mathrm{Spa}(A, A^+)|$ factors over $|U|$, where $U := U(\frac{f_1, \dots, f_n}{g}) \subseteq \mathrm{Spa}(A, A^+)$ and $f_1, \dots, f_n, g \in \pi_0 A$ generate the unit ideal. Then $\mathrm{Spa}(\alpha)$ factors uniquely through U.

Proof. We show (a). We may assume $A \cong \pi_0 A$. Let $R \subseteq A$ be an integrally closed subring. It suffices to show that the inclusion $R \subseteq \overline{R} := \big\{ f \in A \,\big|\, |f(x)| \le 1 \text{ for all } x \in |\mathrm{Spa}(A, R)| \big\}$ is an equality. Assume there is $f \in \overline{R}$ with $f \notin R$. Consider $R[f^{-1}] \subseteq A[f^{-1}]$. Note that $f \notin R[f^{-1}]$, since R is integrally closed and $f \notin R$. Hence, there exists a prime ideal $\mathfrak{p}$ of $R[f^{-1}]$ containing f^{-1}. Let $\mathfrak{q} \subseteq \mathfrak{p}$ be a minimal prime ideal. The injection $R[f^{-1}]_{\mathfrak{q}} \hookrightarrow A[f^{-1}]_{\mathfrak{q}}$ shows $A[f^{-1}]_{\mathfrak{q}} \neq 0$. Hence, there exists a prime ideal $\widetilde{\mathfrak{q}}$ of $A[f^{-1}]$ so that $\widetilde{\mathfrak{q}} \cap R[f^{-1}] \subseteq \mathfrak{q}$; as $\mathfrak{q}$ is minimal, we even have equality. Let K be the fraction field of $A[f^{-1}]/\widetilde{\mathfrak{q}}$ and $K^+ \subseteq K$ a valuation ring dominating $(R[f^{-1}]/\mathfrak{q})_{\mathfrak{p}}$.[5] The canonical map $(A, R) \to (K, K^+)$ sends f into $K \smallsetminus K^+$. The corresponding point $x \in |\mathrm{Spa}(A, R)|$ then satisfies $|f(x)| > 1$, which contradicts $f \in \overline{R}$. This shows $R = \overline{R}$.

For (b), we first show $\alpha(g) \in B^\times$. If $\alpha(g)$ were not invertible, we would find a prime ideal $\mathfrak{p}$ of B containing $\alpha(g)$. Denoting K the fraction field of $B/\mathfrak{p}$, we obtain a point $x \in |\mathrm{Spa}(B, B^+)|$ corresponding to the natural map $(B, B^+) \to (K, K)$ such that $|\alpha(g)(x)| = 0$. But this contradicts the assumption that $|\mathrm{Spa}(B, B^+)|$ maps into $|U|$. Hence, α factors through a map $A[\frac{1}{g}] \to B$. Since also $|\alpha(f_i/g)(x)| \le 1$ for all $x \in |\mathrm{Spa}(B, B^+)|$, we deduce $\alpha(f_i/g) \in B^+$ from (a). Hence α factors through $\big(A\big[\frac{1}{g}\big], A^+\big[\frac{f_1}{g}, \dots, \frac{f_n}{g}\big]\big)$ as desired. ∎

[4]Here, we say $x_1 \colon \mathrm{Spa}(K_1, K_1^+) \to X$ and $x_2 \colon \mathrm{Spa}(K_2, K_2^+) \to X$ are equivalent, $x_1 \sim x_2$, if there exists a valuation ring L^+ with fraction field L and maps $\varphi_i \colon \mathrm{Spa}(L, L^+) \to \mathrm{Spa}(K_i, K_i^+)$ such that $x_1\varphi_1 = x_2\varphi_2$ and $\mathfrak{m}_{L^+} \cap K_i^+ = \mathfrak{m}_{K_i^+}$, for $i = 1, 2$.

[5]See `https://en.wikipedia.org/wiki/Valuation_ring#Dominance_and_integral_closure` or [Sta22, Lemma 00IA] for the existence of K^+.

If $t_1, \dots, t_n \in \pi_0 A$ generate the unit ideal, then it is clear that $\left\{|U(\frac{t_1,\dots,t_n}{t_i})|\right\}_{i=1}^n$ covers $|X_0|$. In fact, we have:

4.3. **Lemma** ([Sch19, Lemma 10.4]). *Let (A, A^+) be a discrete Huber pair and $X = \mathrm{Spa}(A, A^+)$. Assume that $\mathcal{U} = \{U_1, \dots, U_n\}$ is a covering of $|X|$ by rational subsets. Then there exist $s_1, \dots, s_N \in \pi_0 A$ generating the unit ideal such that $\{U(\frac{s_1,\dots,s_N}{s_j})\}_{j=1}^N$ refines $\mathcal{U}$.*

Proof. We may assume $A = \pi_0 A$. First assume that we can write $U_i = U(\frac{T_i}{g_i})$, where $g_i \in T_i$ and $T_i \subseteq A$ is a finite subset generating the unit ideal. Let T be the image of the multiplication map $T_1 \times \dots \times T_n \to A$, and let $S \subseteq T$ be the subset of those elements $t_1 \cdots t_n$ with $t_i = g_i$ for some i. We claim that $\{U(\frac{S}{s})\}_{s \in S}$ is the desired refinement of $\mathcal{U}$. Note that $(S) = (g_1, \dots, g_n)$, because each T_i generates the unit ideal. If $(g_1, \dots, g_n)$ were a proper ideal, it would be contained in a maximal ideal $\mathfrak{m}$ of A. But then the point $\mathrm{Spa}(A/\mathfrak{m}, A/\mathfrak{m}) \to X$ is not contained in any $|U_i|$, which contradicts the fact that $\mathcal{U}$ is a covering of $|X|$. Therefore, S generates the unit ideal in A. Let $s \in S$ be arbitrary. Observe that the obvious inclusion $|U(\frac{T}{s})| \subseteq |U(\frac{S}{s})|$ is an equality: let $x \in |U(\frac{S}{s})|$ and $t \in T$; then $x \in U_i$, for some i, and hence, denoting $s_t \in S$ the element obtained from t by changing the i-th factor to g_i, we deduce $|t(x)| \leq |s_t(x)| \leq |s(x)|$. This proves the equality, and then Proposition 4.2.(a) implies $U(\frac{T}{s}) = U(\frac{S}{s})$. Finally, since s contains some factor g_i, we have $U(\frac{T}{s}) \subseteq U(\frac{T_i}{g_i}) = U_i$.

It remains to show that any point $x \in |U_i|$ closed with respect to the topology generated by the rational domains is contained in a rational domain of the form $|U(\frac{T}{g})| \subseteq |U_i|$, where $g \in T$ and $T \subseteq A$ generates the unit ideal. Write $x\colon (A, A^+) \to (K, K^+)$ and let $\mathfrak{p}$ maximal among the prime ideals of K^+ such that $x(A) \subseteq K^+_{\mathfrak{p}}$. We obtain a new point $y\colon (A, A^+) \to (K^+_{\mathfrak{p}}/\mathfrak{p}K^+_{\mathfrak{p}}, K^+/\mathfrak{p})$. Note that, if $f, g \in A$ are such that y factors through $(A, A^+) \to (A[\frac{1}{g}], A^+[\frac{f}{g}])$, the same is true for x. This means that y is a specialization of x. As x is closed, we deduce $\mathfrak{p} = \{0\}$; in other words: $x(A)$ and K^+ generate K as a ring. Writing $U_i = U\big(\frac{f_1,\dots,f_n}{g}\big)$, we deduce from $x \in |U_i|$ that x extends to a map $A[\frac{1}{g}] \to K$. By what we have just shown, there exists $h \in A$ with $|h(x)| \geq |\frac{1}{g}(x)|$, that is, $|gh(x)| \geq 1$. Hence, $x \in |U\big(\frac{f_1 h,\dots,f_n h,1}{gh}\big)| \subseteq |U_i|$, which is what we wanted to prove. $\square$

4.4. **Proposition.** *A finite family $\{(A, A^+)_\square \to (B_i, B_i^+)_\square\}_i$ of $G_{\omega\text{-adic}}$-localizations is a $G_{\omega\text{-adic}}$-analytic cover if and only if the $\mathcal{U} = \{|\mathrm{Spa}(\pi_0 B_i, B_i^+)|\}_i$ forms a cover of $|\mathrm{Spa}(\pi_0 A, A^+)|$.*

Proof. Assume that $\mathcal{U}$ is a cover. We show that the family of functors

$$M \longmapsto M\big|_{\mathrm{Spa}(B_i, B_i^+)} := M \otimes_{(A,A^+)_\square} (B_i, B_i^+)_\square$$

is conservative. By refining $\mathcal{U}$, we may by Lemma 4.3 assume $\mathrm{Spa}(B_i, B_i^+) = U(\frac{f_1,\dots,f_n}{f_i})$, where $f_1, \dots, f_n \subseteq \pi_0 A$ generate the unit ideal.

Let $M \in \mathcal{D}_\square(A, A^+)$ such that $M \otimes_{(A,A^+)_\square} (B_i, B_i^+)_\square = 0$ for all i. It suffices to show $M[\frac{1}{f_i}] = 0$ for all i. In other words, we may replace (A, A^+) with $(A[\frac{1}{f_i}], A^+[\frac{1}{f_i}])$ and thus assume that $f_i = 1$. For notational convenience, we assume $i = n$. We prove $M = 0$ by induction on n.

For $n = 2$, the covering is given by $|U(\frac{1}{f})| = \{1 \leq |f(x)|\}$ and $|U(\frac{f}{1})| = \{|f(x)| \leq 1\}$. This covering arises by base change from the covering of $|\mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z})|$ by $|\mathrm{Spa}(\mathbb{Z}[t, t^{-1}], \mathbb{Z}[t^{-1}])|$ and $|\mathrm{Spa}(\mathbb{Z}[t], \mathbb{Z}[t])|$. Note that $M \otimes_{(\mathbb{Z}[t],\mathbb{Z})_\square} (\mathbb{Z}[t, t^{-1}], \mathbb{Z}[t^{-1}])_\square = 0$ if and only if M is a $\mathbb{Z}[\![t]\!]$-module and $M \otimes_{(\mathbb{Z}[t],\mathbb{Z})_\square} \mathbb{Z}[t]_\square = 0$ if and only if M is a $\mathbb{Z}((t^{-1}))$-module (Exercise 1.17). Thus, the assumption means that M is a module over $C := \mathbb{Z}[\![t]\!] \otimes_{(\mathbb{Z}[t],\mathbb{Z})_\square} \mathbb{Z}((t^{-1}))$. We claim $C = 0$ to finish the base case. Using the resolution (1.12) of $\mathbb{Z}((t^{-1}))$, we need to see that the top map in the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}[\![t]\!] \otimes_{(\mathbb{Z}[t],\mathbb{Z})_\square} \big(\mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[\![y]\!]\big) \arrow[r, "\mathrm{id} \otimes [\cdot(t\otimes y - 1\otimes 1)]"] \arrow[d, "\cong"'] & \mathbb{Z}[\![t]\!] \otimes_{(\mathbb{Z}[t],\mathbb{Z})_\square} \big(\mathbb{Z}[t] \otimes_{\mathbb{Z}} \mathbb{Z}[\![y]\!]\big) \arrow[d, "\cong"] \\
\mathbb{Z}[\![t]\!] \otimes_{\mathbb{Z}_\square} \mathbb{Z}[\![y]\!] \ \cong\ \mathbb{Z}[\![t, y]\!] \arrow[r, "\cong", "\cdot ty - 1"'] & \mathbb{Z}[\![t, y]\!] \ \cong\ \mathbb{Z}[\![t]\!] \otimes_{\mathbb{Z}_\square} \mathbb{Z}[\![y]\!]
\end{tikzcd}$$

is an isomorphism; but this is clear since all other maps are isomorphisms. This shows $C = 0$ and hence $M = 0$.

Let now $n > 2$. By the base case, it suffices to show $M\big|_{U(\frac{1}{f_1})} \overset{(a)}{\cong} 0 \overset{(b)}{\cong} M\big|_{U(\frac{f_1}{1})}$.

Ad (a): Replacing $M\big|_{U(\frac{1}{f_1})}$ by M and $(A[1/f_1], A^+[1/f_1])$ by (A, A^+), we may assume $f_1^{-1} \in A^+$. For any $i \leq n-1$ we have

$$U_i = U\left(\frac{f_1, \ldots, f_{n-1}, 1}{f_i}\right) = U\left(\frac{f_1, \ldots, f_{n-1}}{f_i}\right) = U\left(\frac{1, f_2/f_1, \ldots, f_{n-1}/f_1}{f_i/f_1}\right),$$

which means that $\{|U_i|\}_{i=1}^{n-1}$ forms a cover. By the induction hypothesis, we deduce $M = 0$.

Ad (b): Replacing $M\big|_{U(\frac{f_1}{1})}$ by M and $(A, A^+[f_1])$ by (A, A^+), we may assume $f_1 \in A^+$. For any $i = 2, \ldots, n$ we have

$$U_i = U\left(\frac{f_1, \ldots, f_{n-1}, 1}{f_i}\right) = U\left(\frac{f_2, \ldots, f_{n-1}, 1}{f_i}\right),$$

which means that $\{|U_i|\}_{i=2}^{n}$ forms a cover. By the induction hypothesis, we deduce $M = 0$.

Thus, we have shown that $M\big|_{U_i} = 0$, for all i, implies $M = 0$.

For the converse, assume that the family $\{\mathrm{Spa}(B_i, B_i^+) \to \mathrm{Spa}(A, A^+)\}_i$ is a $G_{\omega\text{-adic}}$-analytic cover but that $\mathcal{U}$ does not cover $|\mathrm{Spa}(A, A^+)|$. Let K^+ be a valuation ring with fraction field K and let $(A, A^+)_\square \to (K, K^+)_\square$ be a morphism which does not factor through any $(B_i, B_i^+)_\square$. Then, the family $\{(K, K^+)_\square \to (K, K^+)_\square \otimes_{(A,A^+)_\square} (B_i, B_i^+)_\square\}_i$ is a $G_{\omega\text{-adic}}$-analytic cover but not a cover of $|\mathrm{Spa}(K, K^+)|$. We are thus reduced to the case $(A, A^+) = (K, K^+)$. Now, each (B_i, B_i^+) is of the form $(K, K^+[f_1, \ldots, f_n])$ and two such analytic rings are the same if the corresponding $f_1, \ldots, f_n$ generate the same valuation subring of K.

We make the following observation: If $K^+ \subseteq A, B \subseteq K$ are valuation subrings, then $A \subseteq B$ or $B \subseteq A$. Indeed, if $B \subseteq A$, there is nothing to show. Otherwise, take $b \in B \setminus A$. For any $a \in A$ we have $\frac{a}{b} \in K^+$ or $\frac{b}{a} \in K^+$, since K^+ is a valuation ring. But if $\frac{b}{a} \in K^+ \subseteq A$, then also $b = a \cdot \frac{b}{a} \in A$ contradicting the choice of b. Hence, we have $\frac{a}{b} \in K^+ \subseteq B$ and then $a = b \cdot \frac{a}{b} \in B$, that is, $A \subseteq B$.

By the observation, we may assume that the $G_{\omega\text{-adic}}$-analytic cover consists of a single map $(K, K^+) \to (K, K_0^+)$. Now, we have an adjunction

$$- \otimes_{(K,K^+)_\square} (K, K_0^+)_\square \colon \mathcal{D}_\square(K, K^+) \rightleftarrows \mathcal{D}_\square(K, K_0^+) : \mathsf{fgt},$$

where $- \otimes_{(K,K^+)_\square} (K, K_0^+)_\square$ is conservative and fgt is fully faithful. It follows formally that fgt is an equivalence, which shows $(K, K^+)_\square = (K, K_0^+)_\square$. Since the functor $(A, A^+) \mapsto (A, A^+)_\square$ is fully faithful (Proposition 2.10.(a)), we deduce $K^+ = K_0^+$. But then $|\mathrm{Spa}(K, K_0)| = |\mathrm{Spa}(K, K^+)|$, yielding the desired contradiction. $\square$

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