Triangulated and Derived Categories

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Preface

These are the notes for a lecture course on triangulated and derived categories offered in the Winter Term 2024/25 at the University of Paderborn. It is the continuation of a course on homological algebra, and consequenly the course is aimed at students with some basic knowedge of category theory (*e.g.*, Yoneda lemma, (co)limits, adjunctions) and abelian categories. The lecture focuses on the theory rather than examples, and so it helps for the understanding if the reader has had some exposure to classical derived functors. Nevertheless, the aim of these lectures is to give a systematic and largely self-contained treatment of triangulated and derived categories.

Introduction

The evolution of derived functors and the derived category

This section serves an entirely motivational purpose addressed at an audience that is already familiar with the basic notions of homological algebra. We want to trace the history of derived functors and the ideas that led to the definition of the derived category. To keep the exposition reasonably short there will be no proofs, but most of the unproven statements are straightforward suitable exercises for the interested reader (with one exception which is indicated below).

Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor between abelian categories. The non-exactness of F means that we lose information by applying F to exact sequences. To recover the lost information, one can try to define a family of functors $\mathrm{R}^\bullet F = \{\mathrm{R}^i F\colon \mathcal{C} \to \mathcal{D}\}_{i\geq 0}$ such that $\mathrm{R}^0 F \cong F$ and every short exact sequence $0 \to A \to B \to C \to 0$ gives rise to a long exact sequence

$$0 \to F(A) \to F(B) \to F(C) \to \mathrm{R}^1F(A) \to \mathrm{R}^1F(B) \to \mathrm{R}^1F(C) \to \mathrm{R}^2F(A) \to \cdots$$

such that $\mathrm{R}^\bullet F$ is universal in a precise sense (see Definition 0.5 below).

Following [Buc60] or [CE56, Chapter III], let us take a naive approach via "satellites". Let $A \in \mathcal{C}$. Then for every short exact sequence $\mathcal{E} = [0 \to A \xrightarrow{f} P \xrightarrow{g} Q \to 0]$ in $\mathcal{C}$ we can define a new object $S_{\mathcal{E}}F(A) \in \mathcal{D}$ as the cokernel

$$F(P) \xrightarrow{F(g)} F(Q) \longrightarrow S_{\mathcal{E}}F(A) \to 0.$$

Of course, $S_{\mathcal{E}}F(A)$ depends on the choice of P and f. But one easily checks that if we have a morphism

$$\begin{tikzcd} \mathcal{E}\colon & 0 \arrow[r] & A \arrow[r, "f"] \arrow[d, equal] & P \arrow[r, "g"] \arrow[d, "\alpha"] & Q \arrow[r] \arrow[d, "\alpha'"] & 0 \\ \mathcal{E}'\colon & 0 \arrow[r] & A \arrow[r, "f'"'] & P' \arrow[r, "g'"'] & Q' \arrow[r] & 0 \end{tikzcd} \tag{0.1}$$

of short exact sequences (where the left vertical map is the identity on A), we obtain a morphism

$$\theta^{\mathcal{E}}_{\mathcal{E}'}\colon S_{\mathcal{E}}F(A) \to S_{\mathcal{E}'}F(A).$$

Exercise 0.1. Show that $\theta^{\mathcal{E}}_{\mathcal{E}'}$ is independent of α (and α').

We now define the category[1] $\mathcal{I}_A$ whose objects are short exact sequences $\mathcal{E} = [0 \to A \to P \to Q \to 0]$ and where we have a unique morphism $\mathcal{E} \leq \mathcal{E}'$ if and only if there exists a morphism of short exact sequences as in (0.1).

Exercise 0.2. Verify the following statements:

- $\mathcal{I}_A$ is a directed class, *i.e.*, for all $\mathcal{E}, \mathcal{E}' \in \mathcal{I}_A$, there exists $\mathcal{E}'' \in \mathcal{I}_A$ such that $\mathcal{E} \leq \mathcal{E}''$ and $\mathcal{E}' \leq \mathcal{E}''$.

- The assignment $\mathcal{I}_A \to \mathcal{D}$, $\mathcal{E} \mapsto S_{\mathcal{E}}F(A)$ is a functor.

We may thus define

$$SF(A) \coloneqq \varinjlim_{\mathcal{E} \in \mathcal{I}_A} S_{\mathcal{E}}F(A) \in \mathcal{D}$$

whenever the colimit exists.

Exercise 0.3. Let $A \in \mathcal{C}$ and suppose that there exists a monomorphism $A \hookrightarrow I$ into an injective object of $\mathcal{C}$. Show that $\mathcal{E} = [0 \to A \to I \to A/I \to 0]$ is the unique maximal element in $\mathcal{I}_A$. We conclude that the canonical map $S_{\mathcal{E}}F(A) \xrightarrow{\sim} SF(A)$ is an isomorphism in $\mathcal{D}$ (and in particular $SF(A)$ exists).

Exercise 0.4. (i) Show that the assignment $A \mapsto SF(A)$ defines an additive functor $\mathcal{C} \to \mathcal{D}$.

(ii) (Difficult) Suppose that the formation of filtered colimits in $\mathcal{D}$ is exact. Show that for all short exact sequences $0 \to A \to B \to C \to 0$ in $\mathcal{C}$, we obtain an exact sequence

$$F(A) \to F(B) \to F(C) \to SF(A) \to SF(B) \to SF(C).$$

The construction of SF actually makes sense if F is only required to be "half exact", meaning that, if $0 \to A \to B \to C \to 0$ is a short exact sequence, then $F(A) \to F(B) \to F(C)$ is exact. Defining $\mathrm{R}^0F \coloneqq F$ and $\mathrm{R}^nF \coloneqq S(\mathrm{R}^{n-1}F)$ for all $n \geq 1$ then achieves the desired construction of the family $\mathrm{R}^\bullet F$. In fact, the construction shows that $\mathrm{R}^\bullet F$ is a universal δ-functor in the following sense:

Definition 0.5. Let $\mathcal{C}$ and $\mathcal{D}$ be abelian categories. A *δ-functor* $T^\bullet \colon \mathcal{C} \to \mathcal{D}$ consists of the following data:

(i) for every $i \in \mathbb{Z}_{\geq 0}$ a functor $T^i \colon \mathcal{C} \to \mathcal{D}$;

(ii) for every $i \in \mathbb{Z}_{\geq 0}$ and every short exact sequence

$$\mathcal{E} = [0 \to A \to B \to C \to 0] \tag{0.2}$$

in $\mathcal{C}$ a morphism $\delta^i_{\mathcal{E}} \colon T^i(C) \to T^{i+1}(A)$, called the *connecting homomorphism*;

these data are required to satisfy the following conditions:

[1] a partially ordered class, really

(a) for all short exact sequences (0.2) the sequence

$$\begin{tikzcd}
0 \arrow[r] & T^0(A) \arrow[r] & T^0(B) \arrow[r] & T^0(C) \arrow[dll, rounded corners, "\delta^0_{\mathcal{E}}" description] \\
 & T^1(A) \arrow[r] & T^1(B) \arrow[r] & T^1(C) \arrow[dll, rounded corners, "\delta^1_{\mathcal{E}}" description] \\
 & T^2(A) \arrow[r] & T^2(B) \arrow[r] & \cdots
\end{tikzcd}$$

is exact—it is called the *long exact sequence* associated with (0.2);

(b) Naturality: for every morphism

$$\begin{tikzcd}
\mathcal{E}\colon & 0 \arrow[r] & A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & 0 \\
\mathcal{E}'\colon & 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0
\end{tikzcd}$$

of short exact sequences, and for all $i \in \mathbb{Z}_{\geq 0}$, the diagram

$$\begin{tikzcd}
T^i(C) \arrow[r, "\delta^i_{\mathcal{E}}"] \arrow[d] & T^{i+1}(A) \arrow[d] \\
T^i(C') \arrow[r, "\delta^i_{\mathcal{E}'}"'] & T^{i+1}(A')
\end{tikzcd}$$

is commutative.

Moreover, $T^\bullet$ is called *universal* if, additionally, it satisfies the following condition:

(c) Universality: for every δ-functor $\widetilde{T}^\bullet\colon \mathcal{C} \to \mathcal{D}$ and every natural transformation $\alpha^0\colon T^0 \to \widetilde{T}^0$, there exist *unique* natural transformations $\alpha^i\colon T^i \to \widetilde{T}^i$ ($i \geq 0$) such that for every short exact sequence (0.2) and all $i \geq 0$ the diagram

$$\begin{tikzcd}
T^i(C) \arrow[r, "\delta^i_{\mathcal{E}}"] \arrow[d, "\alpha^i_C"'] & T^{i+1}(A) \arrow[d, "\alpha^{i+1}_A"] \\
\widetilde{T}^i(C) \arrow[r, "\delta'^i_{\mathcal{E}}"'] & \widetilde{T}^{i+1}(A)
\end{tikzcd}$$

is commutative.

In other words: a δ-functor $T^\bullet\colon \mathcal{C} \to \mathcal{D}$ is universal if and only if for every other δ-functor $\widetilde{T}^\bullet\colon \mathcal{C} \to \mathcal{D}$, every natural transformation $T^0 \to \widetilde{T}^0$ extends uniquely to a morphism $T^\bullet \to \widetilde{T}^\bullet$ (for the obvious notion of morphism of δ-functors). As usual, it follows that the extension of T^0 to a universal δ-functor is unique up to unique isomorphism.

A useful criterion to check whether a δ-functor is universal is the following:

Exercise 0.6. Recall that an additive functor $T\colon \mathcal{C} \to \mathcal{D}$ is called *effaceable* if for all $A \in \mathcal{C}$ there exists a monomorphism $\iota\colon A \to I$ in $\mathcal{C}$ such that $T(\iota) = 0$.

Let $T^\bullet\colon \mathcal{C} \to \mathcal{D}$ be a δ-functor and suppose that for all $i \geq 1$ the functor T^i is effaceable. Show that $T^\bullet$ is universal.

The above construction shows that $\mathrm{R}^i F(I) = 0$ for all $i \geq 1$, whenever I is injective. Hence, if $\mathcal{C}$ has enough injectives, then $\mathrm{R}^\bullet F$ is indeed a universal δ-functor, which is called the *derived functor* of F. The derived functors $\mathrm{R}^\bullet F$ can be succinctly computed as follows:

Exercise 0.7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor and suppose that $\mathcal{C}$ has enough injectives. Let $A \to I^\bullet$ be an injective resolution of $A \in \mathcal{C}$. Show that $\mathrm{R}^i F(A) \cong \mathrm{H}^i(F(I^\bullet))$ for all $i \geq 0$.

However, this definition of derived functors poses its own set of challenges:

- Exercise 0.7 indicates that the δ-functor $\mathrm{R}^\bullet F$ is actually just a shadow of a more conceptual entity. It suggests that the derived functor should actually be defined as the complex $F(I^\bullet)$, because passing to cohomology and remembering the connecting homomorphisms still loses information. One then needs to construct the correct category of complexes on which derived functors can be defined; this is the *derived category*.

- Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be two left exact functors, and suppose that $\mathcal{B}$ and $\mathcal{C}$ have enough injectives. Can we describe $\mathrm{R}^\bullet(F \circ G)$ in terms of $\mathrm{R}^\bullet F$ and $\mathrm{R}^\bullet G$? In good situations we can recover one from the other, but the answer is still complicated: If for every injective object $I \in \mathcal{B}$ we have $\mathrm{R}^i F(G(I)) = 0$ for all $i \geq 1$, then there is a converging spectral sequence

  $$E_2^{i,j} = \mathrm{R}^i F(\mathrm{R}^j G(B)) \Longrightarrow \mathrm{R}^{i+j}(F \circ G)(B)$$

  for all $B \in \mathcal{B}$; and in even better situations the spectral sequence is manageable. But as soon as we want to compute the derived functor of the composition of *three* left exact functors, we are in trouble. The language of derived categories simplifies the picture.

Keller [Kel96] suggests that the considerations leading Grothendieck to the definition of the derived category of $\mathcal{C}$ are the following: Let $F\colon \mathcal{C} \to \mathcal{D}$ be a left exact functor. If $A \to I^\bullet$ and $A \to J^\bullet$ are two injective resolutions of $A \in \mathcal{C}$, then there is a homotopy equivalence $I^\bullet \xrightarrow{\simeq} J^\bullet$ extending the identity on A. Then $F(I^\bullet) \to F(J^\bullet)$ is a homotopy equivalence, and in particular a quasi-isomorphism. Instead of injective resolutions it is often more practical to compute $\mathrm{R}^\bullet F(A)$ using an F-acyclic resolution $A \to X^\bullet$, that is, a resolution where each X^i satisfies $\mathrm{R}^j F(X^i) = 0$ for all $j \geq 1$. In that case, the identity on A still extends to a quasi-isomorphism $X^\bullet \to I^\bullet$ such that the map $F(X^\bullet) \to F(I^\bullet)$ is a quasi-isomorphism.

These observations suggest to construct the derived category $\mathsf{D}(\mathcal{C})$ from the category $\mathsf{C}(\mathcal{C})$ of complexes by formally inverting all quasi-isomorphisms. The derived functor is then the "universal extension" of F to a functor $\mathrm{R}F\colon \mathsf{D}(\mathcal{C}) \to \mathsf{D}(\mathcal{D})$.

But why should we care about this abstract definition of derived functors? The answer is that this new formalism allows for very simple formulations and proofs of classical results, which in the traditional language would be a complete mess. We illustrate this with the (rather extreme) example of the Künneth relations. Let X and Y be compact topological spaces and R a commutative ring with 1. Let $\mathcal{F} \in \mathsf{Mod}_R(X)$ and $\mathcal{G} \in \mathsf{Mod}_R(Y)$ be sheaves of R-modules on X and Y, respectively. Then we have:

Theorem 0.8 (classical Künneth formula)**.** *(1) Suppose $R = \mathbb{Z}$ and that either $\mathcal{F}$ or $\mathcal{G}$ is torsion-free. Then we have split short exact sequences*

$$0 \longrightarrow \bigoplus_{p+q=n} \mathrm{H}^p(X, \mathcal{F}) \otimes \mathrm{H}^q(Y, \mathcal{G}) \to \mathrm{H}^n(X \times Y, \mathcal{F} \otimes \mathcal{G}) \to \bigoplus_{p+q=n+1} \mathrm{Tor}_1^{\mathbb{Z}}\big(\mathrm{H}^p(X, \mathcal{F}), \mathrm{H}^q(Y, \mathcal{G})\big) \longrightarrow 0$$

(2) In general, there exist two spectral sequences

$$'E_2^{p,q} = \bigoplus_{r+s=q} \mathrm{Tor}_{-p}^R\big(\mathrm{H}^r(X, \mathcal{F}), \mathrm{H}^s(Y, \mathcal{G})\big),$$
$$''E_2^{p,q} = \mathrm{H}^p(X \times Y, \mathcal{T}or_{-q}^R(\mathcal{F}, \mathcal{G})),$$

which have isomorphic abutments.

Compare this with the following version:

Theorem 0.9 (derived Künneth formula)**.** *There is a natural isomorphism in the derived category $\mathsf{D}(R)$ of R-modules*

$$\mathrm{R}\Gamma(X, \mathcal{F}) \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathrm{R}\Gamma(Y, \mathcal{G}) \xrightarrow{\sim} \mathrm{R}\Gamma(X \times Y, \mathcal{F} \overset{\mathrm{L}}{\underset{R}{\otimes}} \mathcal{G}),$$

where $\mathrm{R}\Gamma(X, -)$ denotes the right derived functor of the global sections functor and $\otimes_R^{\mathrm{L}}$ denotes the left derived functor of the tensor product functor.

It turns out that the derived category $\mathsf{D}(\mathcal{C})$ is not abelian in general, but instead carries the structure of a *triangulated category*: It is canonically endowed with a shift functor and "triangles" which provide a formalism akin to that of abelian categories. The idea is that the localization functor $\mathsf{C}(\mathcal{C}) \to \mathsf{D}(\mathcal{C})$ factors through the homotopy category $\mathsf{K}(\mathcal{C})$ of complexes. The advantage of working with the intermediate category $\mathsf{K}(\mathcal{C})$ is (at least) two-fold: for one, $\mathsf{K}(\mathcal{C})$ has the structure of a triangulated category which is inherited by $\mathsf{D}(\mathcal{C})$. On ther other hand, $\mathsf{K}(\mathcal{C})$ admits a "calculus of fractions" for quasi-isomorphisms, and as a consequence it turns out that $\mathsf{D}(\mathcal{C})$ is actually locally small.

The goal of these lectures is to provide a detailed treatment of derived and triangulated categories.

Chapter 1

Recollections

§1. Additive Categories

Although we expect the reader to know what categories and abelian categories are, the treatment of additive categories in the literature is unsatisfactory, which is why we will spend some time on discussing them.

Definition 1.1. Let $\mathcal{C}$ be a category with finite products, and denote by $*$ the terminal object.

(i) A *monoid* in $\mathcal{C}$ is a tuple (M, e, m) consisting of the following data:

   (a) an object $M \in \mathcal{C}$;
   (b) a map $e\colon * \to M$ in $\mathcal{C}$, called the *identity* or *unit*;
   (c) a map $m\colon M \times M \to M$ in $\mathcal{C}$, called the *multiplication*;

these data are required to make the following diagrams commute:

$$\begin{tikzcd} M \times (M \times M) \arrow[r, "\sim"] \arrow[d, "\mathrm{id}\times m"'] & (M \times M) \times M \arrow[r, "m\times\mathrm{id}"] & M \times M \arrow[d, "m"] \\ M \times M \arrow[rr, "m"'] & & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\sim"] \arrow[d, "\sim"'] & * \times M \arrow[r, "e\times\mathrm{id}"] & M \times M \arrow[dd, "m"] \\ M \times * \arrow[d, "\mathrm{id}\times e"'] & & \\ M \times M \arrow[rr, "m"'] & & M \end{tikzcd}$$

The monoid (M, e, m) is called *commutative* if in addition $m \circ s = m$, where $s\colon M \times M \to M \times M$ is the automorphism switching the factors.

A morphism $(M, e, m) \to (M', e', m')$ of monoids consists of a map $f\colon M \to M'$ in $\mathcal{C}$ such that the diagrams

$$\begin{tikzcd} M \times M \arrow[r, "m"] \arrow[d, "f\times f"'] & M \arrow[d, "f"] \\ M' \times M' \arrow[r, "m'"'] & M' \end{tikzcd} \qquad \begin{tikzcd} * \arrow[r, "e"] \arrow[dr, "e'"'] & M \arrow[d, "f"] \\ & M' \end{tikzcd}$$

commute.

We denote by $\mathsf{Mon}(\mathcal{C})$ (resp. $\mathsf{CMon}(\mathcal{C})$) the category of monoids (resp. commutative monoids) in $\mathcal{C}$.

(ii) A monoid (M, e, m) in $\mathcal{C}$ is called a *group* if the map

$$\begin{aligned}(m, \mathrm{pr}_2)\colon M \times M &\xrightarrow{\sim} M \times M,\\ (a, b) &\longmapsto (a \cdot b, b)\end{aligned}$$

is an isomorphism in $\mathcal{C}$. A *commutative group* in $\mathcal{C}$ is a group which is commutative as a monoid.

We define $\mathsf{Grp}(\mathcal{C}) \subseteq \mathsf{Mon}(\mathcal{C})$ as the full subcategory of groups in $\mathcal{C}$. Similarly, we write $\mathsf{CGrp}(\mathcal{C}) \subseteq \mathsf{CMon}(\mathcal{C})$ for the full subcategory of commutative groups in $\mathcal{C}$.

Instead of (M, e, m) we will usually just write M to denote a monoid/group and leave the identity and multiplication implicit.

Example 1.2. (i) A (commutative) monoid in $\mathcal{C} = \mathsf{Set}$ is a (commutative) monoid in the traditional sense, *i.e.*, a tuple $(M, e, \cdot)$ consisting of a set M, an associative (commutative) binary operation $M \times M \to M$, $(a, b) \mapsto ab$ and a two-sided neutral element $e \in M$.

Similarly, a (commutative) group in Set is a (commutative) group in the traditional sense.

(ii) What is a monoid in the category $\mathcal{C} = \mathsf{Ab}$ of abelian groups? We claim that on every $M \in \mathsf{Ab}$ there exists a unique structure of a commutative group in Ab: Since $* = \{0\}$ is also initial, there is a unique map $e\colon \{0\} \to M$ (namely the one with image $0 \in M$). As the product $\times$ is also a coproduct in Ab, the multiplication $m\colon M \times M \to M$ is uniquely determined by its restrictions to $\{0\} \times M$ and $M \times \{0\}$, where it is required to be the identity. But this means $m(a, b) = a + b$ for all $a, b \in M$. So the structure (M, e, m) just encodes the structure of being an abelian group. Moreover, every morphism in Ab is automatically a morphism of commutative groups in Ab.

To summarize, we have $\mathsf{CGrp}(\mathsf{Ab}) = \mathsf{Grp}(\mathsf{Ab}) = \mathsf{CMon}(\mathsf{Ab}) = \mathsf{Mon}(\mathsf{Ab}) = \mathsf{Ab}$.

The last example suggests to axiomatize those categories for which each object is canonically endowed with the structure of a commutative group.

Definition 1.3. Let $\mathcal{C}$ be a category.

(a) A *zero object* is an object $0 \in \mathcal{C}$ which is both inital and final.

In the case where $\mathcal{C}$ admits a zero object 0, we make the following trivial observations:

- Every initial (resp. final) object of $\mathcal{C}$ is a zero object.[1]
- For all $M, N \in \mathcal{C}$ the set $\mathrm{Hom}_{\mathcal{C}}(M, N)$ has a distinguished element given by the unique map $M \to 0 \to N$; it is called the *zero morphism* and is also denoted by 0. Moreover, for every other morphism f in $\mathcal{C}$ we have $0 \circ f = f \circ 0 = 0$.

[1] In the literature, $\mathcal{C}$ is called *pointed* if it admits a zero object.

(b) Assume that $\mathcal{C}$ admits a zero object 0. Let $M, N \in \mathcal{C}$ such that the product $M \times N$ exists. We say that $M \times N$ is a *biproduct*, and write

$$M \oplus N := M \times N,$$

if the canonical maps $i_M : M = M \times 0 \to M \times N$ and $i_N : N = 0 \times N \to M \times N$ exhibit $M \times N$ as a coproduct in $\mathcal{C}$. In other words, $M \sqcup N$ exists and the map $(i_M, i_N) : M \sqcup N \xrightarrow{\sim} M \times N$ is an isomorphism.

We say that $\mathcal{C}$ *has biproducts* if $\mathcal{C}$ admits a zero object and finite products, and every product is a biproduct.

Lemma 1.4. *Let $\mathcal{C}$ be a category with biproducts. Then we have $\mathsf{CMon}(\mathcal{C}) = \mathsf{Mon}(\mathcal{C}) = \mathcal{C}$.*

Proof. The argument is the same as in Example 1.2(ii). □

Definition 1.5. Let $\mathcal{C}$ be a category which has biproducts (and in particular a zero object 0).

(i) $\mathcal{C}$ is called *additive* if every $M \in \mathcal{C}$ is a (commutative) group when endowed with its canonical commutative monoid structure.

(ii) A functor $F : \mathcal{C} \to \mathcal{C}'$ between additive categories is called *additive* if F preserves finite products. Equivalently, the canonical maps $F(M \times N) \xrightarrow{\sim} F(M) \times F(N)$ and $F(*) \xrightarrow{\sim} *$ are isomorphisms, for all $M, N \in \mathcal{C}$.

Example 1.6. (i) We have seen above that Ab is an additive category.

(ii) More generally, for every ring R, the category $\mathsf{Mod}(R)$ of R-modules is additive.

(iii) Let X be a topological space. Then the category $\mathrm{PSh}(X, \mathsf{Ab})$ of presheaves with values in abelian groups is additive. Similarly, the full subcategory $\mathrm{Shv}(X, \mathsf{Ab}) \subseteq \mathrm{PSh}(X, \mathsf{Ab})$ of sheaves with values in Ab is additive.

Lemma 1.7. *Let $F : \mathcal{C} \to \mathcal{D}$ be a functor which preserves finite products. Then F preserves (commutative) monoids/groups. In other words, F induces a functor*

$$\mathsf{Mon}(\mathcal{C}) \longrightarrow \mathsf{Mon}(\mathcal{D})$$

which restricts to functors $\mathsf{CMon}(\mathcal{C}) \to \mathsf{CMon}(\mathcal{D})$, $\mathsf{Grp}(\mathcal{C}) \to \mathsf{Grp}(\mathcal{D})$, and $\mathsf{CGrp}(\mathcal{C}) \to \mathsf{CGrp}(\mathcal{D})$.

Proof. If (M, e, m) is a monoid in $\mathcal{C}$, we obtain a monoid $(F(M), e_{F(M)}, m_{F(M)})$ in $\mathcal{D}$ by setting $e_{F(M)} : * \xrightarrow{\sim} F(*) \xrightarrow{F(e)} F(M)$ and $m_{F(M)} : F(M) \times F(M) \xrightarrow{\sim} F(M \times M) \xrightarrow{F(m)} F(M)$. Since F preserves finite products (and in particular the terminal object) and the definition of "monoid" is entirely diagrammatic, it is straightforward to check that $F(M)$ is a monoid. More precisely, we have the following commutative diagrams:

$$
\begin{tikzcd}
(F(M)\times F(M))\times F(M) \arrow[dd, bend right=60, "m_{F(M)}\times\mathrm{id}"'] & F(M)\times(F(M)\times F(M)) \arrow[l, "\sim"'] \arrow[rr, bend left, "\mathrm{id}\times m_{F(M)}"] & F(M)\times F(M\times M) \arrow[l, "\sim"'] \arrow[r, "\mathrm{id}\times F(m)"] & F(M)\times F(M) \arrow[dd, bend left=60, "m_{F(m)}"] \\
F(M\times M)\times F(M) \arrow[u, "\sim"'] \arrow[d, "F(m)\times\mathrm{id}"] & F((M\times M)\times M) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[d, "F(m\times\mathrm{id})"] & F(M\times(M\times M)) \arrow[u, "\sim"'] \arrow[r, "F(\mathrm{id}\times m)"] & F(M\times M) \arrow[u, "\sim"'] \arrow[d, "F(m)"] \\
F(M)\times F(M) \arrow[rrr, bend right, "m_{F(m)}"'] & F(M\times M) \arrow[l, "\sim"'] \arrow[rr, "F(m)"] & & F(M)
\end{tikzcd}
$$

and

$$
\begin{tikzcd}
F(M) \arrow[r, "\sim"] \arrow[d, "\sim"'] \arrow[dr, equal] & *\times F(M) \arrow[rr, bend left, "e_{F(M)}\times\mathrm{id}"] & F(*)\times F(M) \arrow[l, "\sim"'] \arrow[r, "F(e)\times\mathrm{id}"] & F(M)\times F(M) \\
F(M)\times * \arrow[dd, bend right=60, "\mathrm{id}\times e_{F(M)}"'] & F(M) \arrow[l, "\sim"'] \arrow[r, "\sim"] \arrow[u, "\sim"'] \arrow[d, "\sim"] \arrow[ddrr, equal] & F(*\times M) \arrow[u, "\sim"'] \arrow[r, "F(e\times\mathrm{id})"] & F(M\times M) \arrow[u, "\sim"'] \arrow[dd, "F(m)"'] \arrow[dd, bend left=60, "m_{F(M)}"] \\
F(M)\times F(*) \arrow[u, "\sim"] \arrow[d, "\mathrm{id}\times F(e)"'] & F(M\times *) \arrow[l, "\sim"'] \arrow[d, "F(\mathrm{id}\times e)"] & & \\
F(M)\times F(M) \arrow[rrr, bend right, "m_{F(M)}"'] & F(M\times M) \arrow[l, "\sim"'] \arrow[rr, "F(m)"] & & F(M).
\end{tikzcd}
$$

□

Example 1.8. Let $\mathcal{C}$ be a category with finite products. For every $C \in \mathcal{C}$ the functor

$$\mathrm{Hom}_{\mathcal{C}}(C, \_)\colon \mathcal{C} \to \mathsf{Set}$$

preserves finite products. Hence, Lemma 1.7 shows that for every (commutative) monoid/group M in $\mathcal{C}$, the set $\mathrm{Hom}_{\mathcal{C}}(C, M)$ becomes a (commutative) monoid/group in Set.

The neutral element is the unique map $C \to * \xrightarrow{e} M$. For morphisms $f, g\colon C \to M$ in $\mathcal{C}$, their multiplication $m(f, g)$ is given by the composite $C \xrightarrow{(f,g)} M \times M \xrightarrow{m} M$.

Corollary 1.9. *Let $\mathcal{C}$ be a category with biproducts. Then $\mathcal{C}$ is additive if and only if for all $M, N \in \mathcal{C}$ the commutative monoids $\mathrm{Hom}_{\mathcal{C}}(M, N)$ are abelian groups.*

Moreover, if this is the case, then the composition maps

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}(M, N) \oplus \mathrm{Hom}_{\mathcal{C}}(L, M) &\longrightarrow \mathrm{Hom}_{\mathcal{C}}(L, N), \\ (f, g) &\longmapsto f \circ g \end{aligned}$$

are bilinear.

Proof. The “only if” direction is clear from Example 1.8. Conversely, suppose that all Hom spaces $\mathrm{Hom}_{\mathcal{C}}(M,N)$ are abelian groups. For an arbitrary $N \in \mathcal{C}$ we have to show that the map

$$N \oplus N \xrightarrow{(m,\mathrm{pr}_2)} N \oplus N \tag{1.1}$$

is an isomorphism in $\mathcal{C}$. Since for each $M \in \mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(M,N)$ is an abelian group, the top horizontal map in the diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(M,N) \oplus \mathrm{Hom}_{\mathcal{C}}(M,N) \arrow[r, "(m,\mathrm{pr}_2)", "\sim"'] \arrow[d, equal] & \mathrm{Hom}_{\mathcal{C}}(M,N) \arrow[d, equal] \\ \mathrm{Hom}_{\mathcal{C}}(M,N \oplus N) \arrow[r, "\mathrm{Hom}(M,(m,\mathrm{pr}_2))"'] & \mathrm{Hom}_{\mathcal{C}}(M,N) \end{tikzcd}$$

is an isomorphism. Hence, the lower horizontal map is an isomorphism. By the Yoneda lemma, we conclude that (1.1) is an isomorphism. In other words, N is a group. Hence, $\mathcal{C}$ is additive.

We now prove that the composition maps are bilinear. Since $\mathrm{Hom}_{\mathcal{C}}(L,-)$ enhances to a functor $\mathcal{C} = \mathsf{CGrp}(\mathcal{C}) \to \mathsf{CGrp}(\mathsf{Set}) = \mathsf{Ab}$ by Example 1.8, we deduce that for every morphism $f\colon M \to N$ in $\mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(L,f)\colon \mathrm{Hom}_{\mathcal{C}}(L,M) \to \mathrm{Hom}_{\mathcal{C}}(L,N)$ is additive. In other words, we have $f \circ (g_1 + g_2) = f \circ g_1 + f \circ g_2$ for all $g_1, g_2 \in \mathrm{Hom}_{\mathcal{C}}(L,M)$.

The fact that $(f_1 + f_2) \circ g = f_1 \circ g + f_2 \circ g$ follows from the commutativity of the diagram

$$\begin{tikzcd} & M \arrow[d, "(f_1,f_2)"] \arrow[dr, bend left, "f_1+f_2"] & \\ L \arrow[ur, bend left, "g"] \arrow[r, "(f_1\circ g, f_2\circ g)"] \arrow[rr, bend right, "f_1\circ g+f_2\circ g"'] & N \oplus N \arrow[r, "m"'] & N. \end{tikzcd}$$

□

Exercise 1.10. Let $F\colon \mathcal{A} \to \mathcal{B}$ be a functor between additive categories and assume that F preserves binary products, *i.e.*, $F(M \times N) = F(M) \times F(N)$ for all $M, N \in \mathcal{A}$.

(i) Show that for all $M, N \in \mathcal{C}$ the maps $\mathrm{Hom}_{\mathcal{A}}(M,N) \to \mathrm{Hom}_{\mathcal{B}}(F(M),F(N))$ are additive.

(ii) Let $M \in \mathcal{B}$. Show that $M = 0$ if and only if $\mathrm{id}_M = 0$ in $\mathrm{Hom}_{\mathcal{B}}(M,M)$.

(iii) Show that $F(0) = 0$ and conclude that F is an additive functor.

Remark 1.11. Let $\mathcal{C}$ be a category with biproducts.

(a) Let $M, N \in \mathcal{C}$. Then the biproduct $M \oplus N$ comes with inclusions $i_M = (\mathrm{id}_M, 0)\colon M \hookrightarrow M \oplus N$, $i_N = (0, \mathrm{id}_N)\colon N \hookrightarrow M \oplus N$ and projections $p_M\colon M \oplus N \to M$, $p_N\colon M \oplus N \to N$ such that the following relations are satisfied:

$$\begin{aligned} p_M \circ i_M &= \mathrm{id}_M, & p_N \circ i_N &= \mathrm{id}_N, \\ p_M \circ i_N &= 0, & p_N \circ i_M &= 0, \\ \mathrm{id}_{M\oplus N} &= i_M \circ p_M + i_N \circ p_N. \end{aligned}$$

These data actually characterize $M \oplus N$ as the (bi)product.

Indeed, let $Y \in \mathcal{C}$ and let $\pi_M \colon Y \to M$, $\pi_N \colon Y \to N$ be two maps. We need to show that there exists a unique map $f\colon Y \to M \oplus N$ such that $p_M \circ f = \pi_M$ and $p_N \circ f = \pi_N$. For the uniqueness, let $f\colon Y \to M \oplus N$ be any map with $p_M \circ f = \pi_M$ and $p_N \circ f = \pi_N$. Then

$$\begin{aligned} f &= \mathrm{id}_{M\oplus N} \circ f = (i_M \circ p_M + i_N \circ p_N) \circ f \\ &= i_M \circ p_M \circ f + i_N \circ p_N \circ f = i_M \circ \pi_M + i_N \circ \pi_N, \end{aligned}$$

which proves uniqueness. We now check that $f = i_M \circ \pi_M + i_N \circ \pi_N$ defines a map $Y \to M \oplus N$ with the required properties:

$$\begin{aligned} p_M \circ f &= p_M \circ \big(i_M \circ \pi_M + i_N \circ \pi_N\big) = \pi_M \circ i_M \circ \pi_M + p_M \circ i_N \circ \pi_N \\ &= \mathrm{id}_M \circ \pi_M + 0 \circ \pi_N = \pi_M, \end{aligned}$$

and similarly $p_N \circ f = \pi_N$.

(b) Let $A_1, \dots, A_n, B_1, \dots, B_m \in \mathcal{C}$. Let $i_{A_j} \colon A_j \hookrightarrow \bigoplus_{i=1}^n A_i$, $i_{B_l} \colon B_l \hookrightarrow \bigoplus_{k=1}^m B_k$ be the inclusions and $p_{A_j} \colon \bigoplus_{i=1}^n A_i \twoheadrightarrow A_j$, $p_{B_l} \colon \bigoplus_{k=1}^m B_k \twoheadrightarrow B_l$ the projections characterizing the biproducts as in (a). Then the map

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}}\Big(\bigoplus_{i=1}^n A_i, \bigoplus_{k=1}^m B_k\Big) &\xrightarrow{\sim} \bigoplus_{i=1}^n \bigoplus_{k=1}^m \mathrm{Hom}_{\mathcal{C}}(A_i, B_k), \\ f &\longmapsto \big(p_{B_k} \circ f \circ i_{A_i}\big)_{k,i}, \\ \sum_{i=1}^n \sum_{k=1}^m i_{B_k} \circ f_{ki} \circ p_{A_i} &\longleftarrow (f_{ki})_{k,i} \end{aligned}$$

is an isomorphism of abelian groups. Under this isomorphism, morphisms are composed like matrices. Indeed, given $C_1, \dots, C_r \in \mathcal{C}$ and matrices $(f_{ki})_{k,i} \in \bigoplus_{i=1}^n \bigoplus_{k=1}^m \mathrm{Hom}_{\mathcal{C}}(A_i, B_k)$ and $(g_{lk})_{l,k} \in \bigoplus_{k=1}^m \bigoplus_{l=1}^r \mathrm{Hom}_{\mathcal{C}}(B_k, C_l)$, then we compute the composite as

$$\begin{aligned} (g_{lk})_{l,k} \circ (f_{ki})_{k,i} &\leftrightarrow \Big(\sum_{l=1}^r \sum_{k=1}^m i_{C_l} \circ g_{lk} \circ p_{B_k}\Big) \circ \Big(\sum_{k'=1}^m \sum_{i=1}^n i_{B_{k'}} \circ f_{k'i} \circ p_{A_i}\Big) \\ &= \sum_{l=1}^r \sum_{k=1}^m \sum_{k'=1}^m \sum_{i=1}^n i_{C_l} \circ g_{lk} \circ \underbrace{p_{B_k} \circ i_{B_{k'}}}_{= \mathrm{id}_{B_k} \text{ if } k = k' \text{ and } = 0 \text{ otherwise}} \circ f_{k'i} \circ p_{A_i} \\ &= \sum_{l=1}^r \sum_{i=1}^n i_{C_l} \circ \Big(\sum_{k=1}^m g_{lk} \circ f_{ki}\Big) \circ p_{A_i} \\ &\leftrightarrow \Big(\sum_{k=1}^m g_{lk} \circ f_{ki}\Big)_{l,i}. \end{aligned}$$

For this reason, we usually treat morphisms in an additive category as matrices whenever it is convenient.

We end this section with several examples of how to construct new additive categories from old ones. The verification is left as an exercise for the reader.

Example 1.12. (i) Let $\mathcal{A}$ be an additive category and $\mathcal{A}' \subseteq \mathcal{A}$ a full subcategory which is closed under biproducts. Then $\mathcal{A}'$ is additive.

(ii) If $\mathcal{A}$ is an additive category, then so is its opposite category $\mathcal{A}^{\mathrm{op}}$. (Hint: Use Corollary 1.9.)

(iii) If $\mathcal{A}$ and $\mathcal{B}$ are additive categories, then so is $\mathcal{A} \times \mathcal{B}$.

(iv) If $\mathcal{C}$ is a category and $\mathcal{A}$ is an additive category, then the functor category $\mathrm{Fun}(\mathcal{C}, \mathcal{A})$ is additive.

§2. The Homotopy Category of Complexes

In view of homological algebra, the most important categories associated with an additive category are the category of complexes and its homotopy category.

Definition 2.1. Let $\mathcal{A}$ be an additive category. A *complex* in $\mathcal{A}$ is a tuple $(A^\bullet, d^\bullet)$ consisting of objects $A^i \in \mathcal{A}$ and morphisms $d^i \colon A^i \to A^{i+1}$ ($i \in \mathbb{Z}$) such that $d^{i+1} \circ d^i = 0$ for all $i \in \mathbb{Z}$. The maps d^i are called *differentials* or *boundary maps*. We usually denote a complex by $A^\bullet$ instead of $(A^\bullet, d^\bullet)$ and depict it as

$$A^\bullet = [\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \xrightarrow{d^{i+1}} \cdots].$$

A morphism of complexes $f \colon A^\bullet \to B^\bullet$ consists of a collection of maps $\{f^i \colon A^i \to B^i\}_{i\in\mathbb{Z}}$ such that $d^i_B \circ f^i = f^{i+1} \circ d^i_A$ for all $i \in \mathbb{Z}$.

We denote the category of complexes in $\mathcal{A}$ by $\mathsf{C}(\mathcal{A})$.

Lemma 2.2. *Let $\mathcal{A}$ be an additive category.*

(a) The category $\mathsf{C}(\mathcal{A})$ is additive.

(b) For every $n \in \mathbb{Z}$ the shift functor

$$\begin{aligned}[n] \colon \mathsf{C}(\mathcal{A}) &\longrightarrow \mathsf{C}(\mathcal{A}),\\ A &\longmapsto A[n],\end{aligned}$$

given by $A[n]^i := A^{i+n}$ and $d^i_{A[n]} = (-1)^n d^{i+n}_A$ (for all $i \in \mathbb{Z}$), is an additive automorphism.[2] *We have $[m+n] = [m] \circ [n] = [n] \circ [m]$ for all $m, n \in \mathbb{Z}$.*

(c) For every additive functor $F \colon \mathcal{A} \to \mathcal{B}$ the induced functor $\mathsf{C}(F) \colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{B})$ is additive.

Proof. Straightforward (observe that biproducts in $\mathsf{C}(\mathcal{A})$ are given componentwise and the group condition can be checked on each component separately). ☐

Variant 2.3. The following full additive subcategories of $\mathsf{C}(\mathcal{A})$ are also frequently used:

- the category $\mathsf{C}^+(\mathcal{A})$ of *left bounded complexes*, *i.e.*, complexes $A^\bullet$ for which there exists $i_0 \in \mathbb{Z}$ with $A^i = 0$ for all $i < i_0$:

$$A^\bullet = [\cdots \to 0 \to 0 \to A^{i_0} \to A^{i_0+1} \to A^{i_0+2} \to \cdots]$$

[2] Pictorially speaking, the shift functor [1] shifts a complex one space to the *left*.

- the category $\mathsf{C}^-(\mathcal{A})$ of *right bounded complexes*, *i.e.*, complexes $A^\bullet$ for which there exists $i_0 \in \mathbb{Z}$ with $A^i = 0$ for all $i > i_0$:

$$A^\bullet = [\cdots \to A^{i_0-2} \to A^{i_0-1} \to A^{i_0} \to 0 \to 0 \to \cdots].$$

- the category $\mathsf{C}^b(\mathcal{A}) = \mathsf{C}^+(\mathcal{A}) \cap \mathsf{C}^-(\mathcal{A})$ of *bounded complexes*.

Note that each superscript $* \in \{-, +, b\}$, the statements of Lemma 2.2 apply to $\mathsf{C}^*(\mathcal{A})$.

Definition 2.4. Let $\mathcal{A}$ be an additive category. A morphism $f\colon A^\bullet \to B^\bullet$ in $\mathsf{C}(\mathcal{A})$ is called *null homotopic*, written $f \simeq 0$, if there exists a collection of morphisms $s = \{s^i\colon A^i \to B^{i-1}\}_{i\in\mathbb{Z}}$ in $\mathcal{A}$ such that

$$f^i = s^{i+1}d_A^i + d_B^{i-1}s^i$$

for all $i \in \mathbb{Z}$; in this case s is called a *null homotopy* of f:

$$\begin{tikzcd}
\cdots \arrow[r] & A^{i-1} \arrow[r, "d^{i-1}"] \arrow[d, "f^{i-1}"'] & A^i \arrow[r, "d^i"] \arrow[d, "f^i"] \arrow[dl, "s^i"'] & A^{i+1} \arrow[r, "d^{i+1}"] \arrow[d, "f^{i+1}"] \arrow[dl, "s^{i+1}"] & \cdots \\
\cdots \arrow[r] & B^{i-1} \arrow[r, "d^{i-1}"'] & B^i \arrow[r, "d^i"'] & B^{i+1} \arrow[r] & \cdots
\end{tikzcd}$$

A *homotopy* between two morphisms $f, g\colon A^\bullet \to B^\bullet$ is a null homotopy for $f - g$.

A complex $A^\bullet$ is called *contractible* if $\mathrm{id}_{A^\bullet}$ is null homotopic.

Warning 2.5. Beware that a homotopy $\{s^i\colon A^i \to B^{i-1}\}_{i\in\mathbb{Z}}$ does generally *not* constitute a morphism of complexes $A[1] \to B$ unless it is a null homotopy of the zero morphism!

Exercise 2.6. Let $\mathcal{A}$ be an additive category. Show that a complex $0 \to A^0 \to A^1 \to A^2 \to 0$ is contractible if and only if it is isomorphic to the complex $0 \to A^0 \to A^0 \oplus A^2 \to A^2 \to 0$, where the maps are the obvious ones.

Exercise 2.7. Let $\mathcal{A}$ be an additive category and $(A^\bullet, d)$ a complex.

(a) Show that $d\colon A^\bullet \to A^\bullet[1]$ is a morphism of complexes which is null homotopic.

(b) Suppose there exists a family $\{s^i, t^i\colon A^i \to A^{i-1}\}_i$ of morphisms in $\mathcal{A}$ such that $\mathrm{id}_{A^i} = s^{i-1}d^i + d^{i-1}t^i$ for all $i \in \mathbb{Z}$. Show that $A^\bullet$ is contractible. (Hint: consider the maps $s^i d^{i-1} t^i$.)

Lemma 2.8. *Let $\mathcal{A}$ be an additive category.*

(a) If $f, g\colon A^\bullet \to B^\bullet$ are null homotopic maps of complexes, then so is $f - g$. In other words, the subset $B^0(A^\bullet, B^\bullet) \subseteq \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(A^\bullet, B^\bullet)$ of null homotopic maps is a subgroup.

(b) Let $f\colon A^\bullet \to B^\bullet$ and $g\colon B^\bullet \to C^\bullet$ be morphisms of complexes. If one of f or g is null homotopic, then so is $g \circ f$.

Proof. We prove (a). Let s be a null homotopy for f and t a null homotopy for g. Then the family $\{s^i - t^i\}_{i\in\mathbb{Z}}$ is a null homotopy for $f - g$. Moreover, the zero maps $\{0\colon A^i \to B^{i-1}\}$ are a null homotopy for the zero map $0\colon A^\bullet \to B^\bullet$. Hence, $B^0(A^\bullet, B^\bullet)$ is a subgroup of $\mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(A^\bullet, B^\bullet)$.

For part (b), let s be a null homotopy for f. Then $\{g^{i-1} \circ s^i\}_i$ is a null homotopy for $g \circ f$, since

$$g^i f^i = g^i s^{i+1} d_A^i + g^i d_B^{i-1} s^i = g^i s^{i+1} d_A^i + d_C^{i-1} g^{i-1} s^i.$$

The other case is similar. □

Definition 2.9. Let $\mathcal{A}$ be an additive category. The *homotopy category* of $\mathcal{A}$ is the category $\mathsf{K}(\mathcal{A})$ which has the same objects as $\mathsf{C}(\mathcal{A})$ and with morphism spaces given by the factor groups

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet) \coloneqq \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(A^\bullet, B^\bullet)/B^0(A^\bullet, B^\bullet).$$

The composition is given as follows: Let $f\colon A^\bullet \to B^\bullet$ and $g\colon B^\bullet \to C^\bullet$ be morphisms of complexes representing maps $[f]\colon A^\bullet \to B^\bullet$ and $[g]\colon B^\bullet \to C^\bullet$ in $\mathsf{K}(\mathcal{A})$. Then we put

$$[g] \circ [f] \coloneqq [g \circ f].$$

The definition is independent of the choices of representatives of $[f]$ and $[g]$: Every other representative of $[f]$ is of the form $f + f'$ for some null homotopic map $f'\colon A^\bullet \to B^\bullet$, and similarly every other representative of $[g]$ is given by $g + g'$ for some null homotopic map $g'\colon B^\bullet \to C^\bullet$. Then we have

$$(f + f') \circ (g + g') = f \circ g + f' \circ (g + g') + f \circ g',$$

where $f' \circ (g + g') + f \circ g'$ is null homotopic by Lemma 2.8. Hence $[(f + f') \circ (g + g')] = [f \circ g]$. It is trivial to check that composition on $\mathsf{K}(\mathcal{A})$ is associative and that $[\mathrm{id}_{A^\bullet}]$ is the identity on $A^\bullet \in \mathsf{K}(\mathcal{A})$.

Proposition 2.10. *Let $\mathcal{A}$ be an additive category.*

(i) The category $\mathsf{K}(\mathcal{A})$ and the quotient functor $\mathsf{C}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ are additive.

(ii) For every $n \in \mathbb{Z}$, the shift functor $[n]$ on $\mathsf{C}(\mathcal{A})$ descends to additive automorphism

$$[n]\colon \mathsf{K}(\mathcal{A}) \longrightarrow \mathsf{K}(\mathcal{A}).$$

Proof. Since automorphisms preserve products, part (ii) is obvious. In order to prove (i), we first show that $\mathsf{K}(\mathcal{A})$ admits biproducts and that the quotient functor $Q\colon \mathsf{C}(\mathcal{A}) \to \mathsf{K}(\mathcal{A})$ preserves them.

Note that $\mathsf{K}(\mathcal{A})$ is enriched in Ab, *i.e.*, the sets $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(A^\bullet, B^\bullet)$ are abelian groups and the composition maps are bilinear. Moreover, Q is linear on Hom groups. Now, given $A^\bullet, B^\bullet \in \mathsf{C}(\mathcal{A})$, the biproduct is characterized by the data $(A^\bullet \oplus B^\bullet, i_A, i_B, p_A, p_B)$ satisfying $p_A i_A = \mathrm{id}_A$, $p_B i_B = \mathrm{id}_B$, $p_A i_B = 0 = p_B i_A$ and $\mathrm{id}_{A\oplus B} = i_A p_A + i_B p_B$ (see Remark 1.11). Since Q preserves these conditions, it follows that $Q(A^\bullet) \oplus Q(B^\bullet)$ is the biproduct in $\mathsf{K}(\mathcal{A})$. Moreover, the zero complex is clearly a zero object. It follows that $\mathsf{K}(\mathcal{A})$ and Q are additive. □

Definition 2.11. Let $\mathcal{A}$ be an additive category and $f\colon A^\bullet \to B^\bullet$ a morphism of complexes. We define the *mapping cone* of f as the complex $\mathrm{Mc}(f)$ with terms

$$\mathrm{Mc}(f)^i = A^{i+1} \oplus B^i$$

and differential $d^i_{\mathrm{Mc}(f)} = \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}$.

Example 2.12. Let $\mathcal{A}$ be an abelian category (see Section §3). Let $f\colon A \to B$ be a morphism in $\mathcal{A}$, viewed as a morphism of complexes. Then $\mathrm{Mc}(f)$ is the complex $[0 \to A \xrightarrow{f} B \to 0]$, where B sits in degree 0. Note that

$$\mathrm{H}^{-1}(\mathrm{Mc}(f)) = \mathrm{Ker}(f) \qquad \text{and} \qquad \mathrm{H}^{0}(\mathrm{Mc}(f)) = \mathrm{Coker}(f).$$

See Definition 3.5 for the definition of the cohomology $\mathrm{H}^i(\mathrm{Mc}(f))$.

Proposition 2.13. *Let $\mathcal{A}$ be an additive category.*

(i) Let $A^\bullet \in \mathsf{C}(\mathcal{A})$. Then $\mathrm{Mc}(\mathrm{id}_{A^\bullet})$ is contractible.

(ii) Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes. Then f fits into a sequence of morphisms

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\binom{0}{\mathrm{id}}} \mathrm{Mc}(f)^\bullet \xrightarrow{(\mathrm{id},0)} A[1]^\bullet.$$

which is exact at $\mathrm{Mc}(f)^\bullet$.

(iii) Consider a diagram in $\mathsf{C}(\mathcal{A})$ with solid arrows

$$\begin{tikzcd}
A^\bullet \arrow[r, "f"] \arrow[d, "\alpha"'] & B^\bullet \arrow[r, "\iota_f"] \arrow[d, "\beta"] & \mathrm{Mc}(f)^\bullet \arrow[r, "p_f"] \arrow[d, dashed, "\varphi"] & A[1]^\bullet \arrow[d, "\alpha[1]"] \\
C^\bullet \arrow[r, "g"'] \arrow[ur, Rightarrow, "s"] & D^\bullet \arrow[r, "\iota_g"'] & \mathrm{Mc}(g)^\bullet \arrow[r, "p_g"'] & C^\bullet
\end{tikzcd}$$

and let $s = \{s^i\colon A^i \to D^{i-1}\}_i$ be a null homotopy of $\beta f - g\alpha$. Then there exists a morphism $\varphi = \varphi(\alpha, \beta, s)\colon \mathrm{Mc}(f) \to \mathrm{Mc}(g)$ making the middle and right square commutative in $\mathsf{C}(\mathcal{A})$.

Proof. Let us prove (i). We have to produce a null homotopy of the identity on $\mathrm{Mc}(\mathrm{id}_{A^\bullet})$. To this end, we define

$$s^i \coloneqq \begin{pmatrix} 0 & \mathrm{id}_{A^i} \\ 0 & 0 \end{pmatrix}\colon \mathrm{Mc}(\mathrm{id}_{A^\bullet})^i = A^{i+1} \oplus A^i \longrightarrow A^i \oplus A^{i-1} = \mathrm{Mc}(\mathrm{id}_{A^\bullet})^{i-1}.$$

Now, compute

$$\begin{aligned} s^{i+1}d^i + d^{i-1}s^i &= \begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} -d & 0 \\ \mathrm{id} & d \end{pmatrix} + \begin{pmatrix} -d & 0 \\ \mathrm{id} & d \end{pmatrix}\begin{pmatrix} 0 & \mathrm{id} \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & d \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -d \\ 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\mathrm{Mc}(\mathrm{id})^i}. \end{aligned}$$

Hence, $\mathrm{Mc}(\mathrm{id}_{A^\bullet})$ is indeed contractible. In particular, we have $\mathrm{Mc}(\mathrm{id}_{A^\bullet}) = 0$ in $\mathsf{K}(\mathcal{A})$ and hence Lemma 3.7 shows that $\mathrm{Mc}(\mathrm{id}_{A^\bullet})$ is acyclic.

The fact that the maps $B^\bullet \to \mathrm{Mc}(f)^\bullet$ and $\mathrm{Mc}(f)^\bullet \to A[1]^\bullet$ are morphisms follows from the following straightforward computation:

$$\begin{gathered} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix}\begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} - \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} \circ d = \begin{pmatrix} 0 \\ d_B \end{pmatrix} - \begin{pmatrix} 0 \\ d_B \end{pmatrix} = 0 \\ d \circ (\mathrm{id}, 0) - (\mathrm{id}, 0)\begin{pmatrix} d_{A[1]} & 0 \\ f & d_B \end{pmatrix} = (d_{A[1]}, 0) - (d_{A[1]}, 0) = 0. \end{gathered}$$

The exactness at $\mathrm{Mc}(f)^\bullet$ is obvious.

For part (iii), we consider the map $\varphi\colon \mathrm{Mc}(f)^\bullet \to \mathrm{Mc}(g)^\bullet$ given by

$$\varphi^i = \begin{pmatrix} \alpha^{i+1} & 0 \\ s^{i+1} & \beta^i \end{pmatrix} \colon \mathrm{Mc}(f)^i = A^{i+1} \oplus B^i \to C^{i+1} \oplus D^i = \mathrm{Mc}(g)^i.$$

The computation

$$\begin{aligned}
\varphi \circ d_{\mathrm{Mc}(f)} - d_{\mathrm{Mc}(g)} \circ \varphi &= \begin{pmatrix} \alpha & 0 \\ s & \beta \end{pmatrix} \begin{pmatrix} -d_A & 0 \\ f & d_B \end{pmatrix} - \begin{pmatrix} -d_C & 0 \\ g & d_D \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ s & \beta \end{pmatrix} \\
&= \begin{pmatrix} -\alpha d_A & 0 \\ -s d_A + \beta f & \beta d_B \end{pmatrix} - \begin{pmatrix} -d_C \alpha & 0 \\ g\alpha + d_D s & d_D \beta \end{pmatrix} \\
&= \begin{pmatrix} d_C\alpha - \alpha d_A & 0 \\ \beta f - g\alpha - s d_A - d_D s & \beta d_B - d_D \beta \end{pmatrix} = 0
\end{aligned}$$

shows that φ is a morphism of complexes. It is clear that the middle and right square in the assertion commute. □

Exercise 2.14. Let $\mathcal{A}$ be an additive category and let $f\colon A^\bullet \to B^\bullet$ be a morphism in $\mathsf{C}(\mathcal{A})$.

(a) Show that the composite $A^\bullet \xrightarrow{f} B^\bullet \hookrightarrow \mathrm{Mc}(f)^\bullet$ is null homotopic.

(b) Let $g\colon A^\bullet \to B^\bullet$ be another morphism which is homotopic to f. Construct an isomorphism $\mathrm{Mc}(f) \xrightarrow{\sim} \mathrm{Mc}(g)$ in $\mathsf{C}(\mathcal{A})$.

(c) Show that f is null homotopic if and only if f factors through $\iota_A\colon A^\bullet \to \mathrm{Mc}(\mathrm{id}_{A^\bullet})$.

§3. Abelian Categories

Notation 3.1. Let $\mathcal{A}$ be an additive category in which every morphism admits a kernel and a cokernel. For a morphism $f\colon A \to B$ in $\mathcal{A}$ we put

$$\begin{aligned}
\mathrm{Coim}(f) &:= A/\operatorname{Ker}(f) := \operatorname{Coker}\big(\operatorname{Ker}(f) \hookrightarrow A\big) && \text{(coimage)} \\
\mathrm{Im}(f) &:= \operatorname{Ker}\big(B \twoheadrightarrow \operatorname{Coker}(f)\big) && \text{(image).}
\end{aligned}$$

The map f factors as follows:

$$\begin{tikzcd}
\operatorname{Ker}(f) \arrow[r, hook] & A \arrow[r, "f"] \arrow[d, two heads, "p"'] & B \arrow[r, two heads] & \operatorname{Coker}(f) \\
& \mathrm{Coim}(f) \arrow[r, "\overline{f}"'] & \mathrm{Im}(f). \arrow[u, hook, "i"'] &
\end{tikzcd}$$

The map f is called *strict* if $\overline{f}$ is an isomorphism.

Definition 3.2. An additive category $\mathcal{A}$ is called *abelian* if every morphism is strict and admits a kernel and a cokernel.

Example 3.3. (i) The categories Ab, $\mathsf{Mod}(R)$ (for a ring R), $\mathrm{PSh}(X, \mathsf{Ab})$ and $\mathrm{Shv}(X, \mathsf{Ab})$ (for a topological space X) are abelian categories.

(ii) A category $\mathcal{A}$ is abelian if and only if its opposite category $\mathcal{A}^{\mathrm{op}}$ is abelian.

(iii) If $\mathcal{A}$ is an abelian category, then so is $\mathsf{C}(\mathcal{A})$ (check that kernels and cokernels are given componentwise).

(iv) If $\mathcal{A}$ is abelian, the homotopy category $\mathsf{K}(\mathcal{A})$ is generally not abelian. Concretely, will show in Example 5.5 that $\mathsf{K}(\mathsf{Ab})$ is not abelian.

We collect the following basic properties of abelian categories:

Theorem 3.4. *Let $\mathcal{A}$ be an abelian category.*

(1) Every monomorphism is a kernel (of its cokernel) and every epimorphism is a cokernel (of its kernel).

(2) A morphism in $\mathcal{A}$ is an isomorphism if and only if it is both monic and epic.

(3) Every morphism $f\colon A \to B$ factors uniquely (up to unique isomorphism) as $A \xrightarrow{p} I \xrightarrow{i} B$, where p is an epimorphism and i is a monomorphism.

(4) $\mathcal{A}$ admits finite limits and finite colimits.

One of the main reasons abelian categories were introduced was to do homological algebra.

Definition 3.5. Let $\mathcal{A}$ be an abelian category and let

$$A^\bullet = [\cdots \to A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \to \cdots]$$

be a complex. Note that, since $d^i \circ d^{i-1} = 0$, we always have an inclusion $\mathrm{Im}(d^{i-1}) \subseteq \mathrm{Ker}(d^i)$. We define the *$i$-th cohomology* of $A^\bullet$ as the quotient

$$\mathrm{H}^i(A^\bullet) \coloneqq \mathrm{Ker}(d^i)/\,\mathrm{Im}(d^{i-1}) \in \mathcal{A}.$$

If $\mathrm{H}^i(A^\bullet) = 0$ (equivalently, $\mathrm{Im}(d^{i-1}) = \mathrm{Ker}(d^i)$), we call $A^\bullet$ *exact* at A^i. If $\mathrm{H}^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$, we say that $A^\bullet$ is *acyclic* or *exact*.

Example 3.6. Let $\mathcal{A}$ be an abelian category and let $f\colon A \to B$ be a morphism.

- The sequence $0 \to A \xrightarrow{f} B$ is exact if and only if $\mathrm{Ker}(f) = 0$, *i.e.*, f is monic.
- The sequence $A \xrightarrow{f} B \to 0$ is exact if and only if $\mathrm{Coker}(f) = 0$ (equivalently: $\mathrm{Im}(f) = B$), *i.e.*, f is epic.

Lemma 3.7. *Let $\mathcal{A}$ be an abelian category. For every $i \in \mathbb{Z}$ passing to cohomology induces additive functors $\mathrm{H}^i\colon \mathsf{C}(\mathcal{A}) \to \mathcal{A}$, which factor through $\mathsf{K}(\mathcal{A})$. In particular, contractible complexes are acyclic.*

Proof. Let $f\colon A^\bullet \to B^\bullet$ be a morphism of complexes. From the identity $f^{i+1}d_A^i = d_B^i f^i$ we deduce $f^i(\mathrm{Ker}(d_A^i)) \subseteq \mathrm{Ker}(d_B^i)$ and $f^{i+1}(\mathrm{Im}(d_A^i)) \subseteq \mathrm{Im}(d_B^i)$ for all $i \in \mathbb{Z}$. Hence, we obtain an induced map

$$\mathrm{H}^i(f)\colon \mathrm{H}^i(A^\bullet) = \mathrm{Ker}(d_A^i)/\,\mathrm{Im}(d_A^{i-1}) \to \mathrm{Ker}(d_B^i)/\,\mathrm{Im}(d_B^{i-1}) = \mathrm{H}^i(B^\bullet).$$

It is clear from the construction that $\mathrm{H}^i(\mathrm{id}_{A^\bullet}) = \mathrm{id}_{\mathrm{H}^i(A^\bullet)}$ and that if f, g are composable maps of complexes, then $\mathrm{H}^i(f \circ g) = \mathrm{H}^i(f) \circ \mathrm{H}^i(g)$.

We now check that H^i is additive. Given $A^\bullet, B^\bullet \in \mathsf{C}(\mathcal{A})$, we observe that the canonical maps $\mathrm{Ker}(d^i_{A\oplus B}) \xrightarrow{\sim} \mathrm{Ker}(d^i_A) \oplus \mathrm{Ker}(d^i_B)$ and $\mathrm{Im}(d^{i-1}_{A\oplus B}) \xrightarrow{\sim} \mathrm{Im}(d^{i-1}_A) \oplus \mathrm{Im}(d^{i-1}_B)$ are isomorphisms, because the differential is given componentwise. We obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Im}(d^{i-1}_{A\oplus B}) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Ker}(d^i_{A\oplus B}) \arrow[r] \arrow[d, "\sim"] & \mathrm{H}^i(A \oplus B) \arrow[r] \arrow[d, dashed] & 0 \\
\mathrm{Im}(d^{i-1}_A) \oplus \mathrm{Im}(d^{i-1}_B) \arrow[r] & \mathrm{Ker}(d^i_A) \oplus \mathrm{Ker}(d^i_B) \arrow[r] & \mathrm{H}^i(A) \oplus \mathrm{H}^i(B) \arrow[r] & 0,
\end{tikzcd}$$

where the rows are exact. From the five lemma we conclude that the dashed arrow is an isomorphism. Hence H^i is additive.

To see that H^i factors through $\mathsf{K}(\mathcal{A})$ it suffices to show that if $f \colon A^\bullet \to B^\bullet$ is null homotopic, then $\mathrm{H}^i(f) = 0$. Indeed, let s be a null homotopy of f, so that $f^i = s^{i+1}d^i_A + d^{i-1}_B s^i$ for all i. The identity shows $f^i(\mathrm{Ker}(d^i_A)) \subseteq \mathrm{Im}(d^{i-1}_B)$ and hence $\mathrm{H}^i(f)$ is the zero map. □

Definition 3.8. Let $\mathcal{A}$ be an abelian category. A morphism $f \colon A^\bullet \to B^\bullet$ of complexes is called a *quasi-isomorphism* if the induced map $\mathrm{H}^i(f) \colon \mathrm{H}^i(A^\bullet) \to \mathrm{H}^i(B^\bullet)$ is an isomorphism for all $i \in \mathbb{Z}$.

Exercise 3.9. Let $\mathcal{A}$ be an abelian category. Let $f \colon A^\bullet \to B^\bullet$ be a *homotopy equivalence*, *i.e.*, a morphism of complexes whose image in $\mathsf{K}(\mathcal{A})$ becomes an isomorphism. Show that f is a quasi-isomorphism. Does the converse hold true?

Proposition 3.10. *Let $\mathcal{A}$ be an abelian category and consider a commutative square*

(3.1)
$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "g"'] & B \arrow[d, "g'"] \\
A' \arrow[r, "f'"'] & B'
\end{tikzcd}$$

in $\mathcal{A}$.

(i) The square (3.1) *is a pullback if and only if the sequence* $0 \to A \xrightarrow{\left(\begin{smallmatrix} f \\ -g \end{smallmatrix}\right)} B \oplus A' \xrightarrow{(g', f')} B'$ *is exact. If this is the case, then the induced map* $\mathrm{Ker}(f) \xrightarrow{\sim} \mathrm{Ker}(f')$ *is an isomorphism.*

(ii) The square (3.1) *is a pushout if and only if the sequence* $A \xrightarrow{\left(\begin{smallmatrix} f \\ -g \end{smallmatrix}\right)} B \oplus A' \xrightarrow{(g', f')} B' \to 0$ *is exact. If this is the case, then the induced map* $\mathrm{Coker}(f) \xrightarrow{\sim} \mathrm{Coker}(f')$ *is an isomorphism.*

Addendum: *In (i) the induced map* $\mathrm{Coker}(f) \hookrightarrow \mathrm{Coker}(f')$ *is a monomorphism. In (ii) the induced map* $\mathrm{Ker}(f) \twoheadrightarrow \mathrm{Ker}(f')$ *is an epimorphism.*

Proof. We only prove (i) because (ii) is dual. The first statement is obvious by comparing the universal properties of the pullback and the kernel. For example, if the square is a pullback and $C \xrightarrow{\left(\begin{smallmatrix} b \\ -a \end{smallmatrix}\right)} B \oplus A'$ is a map such that $g'b - f'a = 0$, then by the universal property of the pullback there exists a unique map $C \xrightarrow{c} A$ such that $fc = b$ and $gc = a$, so that $C \xrightarrow{\left(\begin{smallmatrix} b \\ -a \end{smallmatrix}\right)} B \oplus A'$ factors as $C \xrightarrow{c} A \xrightarrow{\left(\begin{smallmatrix} f \\ -g \end{smallmatrix}\right)} B \oplus A'$ as desired. Next, we compute

$$\mathrm{Ker}(f) = A \times_B 0 = (A' \times_{B'} B) \times_B 0 \xrightarrow{\sim} A' \times_{B'} 0 = \mathrm{Ker}(f').$$

We finally prove that $\iota\colon \mathrm{Coker}(f) \hookrightarrow \mathrm{Coker}(f')$ is a monomorphism. We will freely use part (ii). Let $h\colon C \to \mathrm{Coker}(f)$ be a map such that $\iota \circ h = 0$. We need to show $h = 0$. Consider now the pullback square

$$\begin{tikzcd}
C' \arrow[r, two heads, "\pi'"] \arrow[d, "h'"'] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, "h"] \\
B \arrow[r, two heads, "\pi"'] & \mathrm{Coker}(f).
\end{tikzcd}$$

Note that, since π is an epimorphism, the square is a pushout by (ii) and hence also π' is an epimorphism. It therefore suffices to prove $h\pi' = 0$. As $\iota h\pi' = 0$, we deduce that $g'h'\colon C' \to B'$ factors as $C' \xrightarrow{\overline{h}'} \mathrm{Im}(f') \xrightarrow{j} B'$. Consider now the pullback

$$\begin{tikzcd}
C'' \arrow[r, two heads, "\pi''"] \arrow[d, "h''"'] & C' \arrow[d, "\overline{h}'"] \\
A' \arrow[r, two heads, "\overline{f}'"'] & \mathrm{Im}(f').
\end{tikzcd}$$

As $\overline{f}'$ is an epimorphism, we conclude from (ii) that the square is also a pushout and hence π'' is an epimorphism. It therefore suffices to prove $h\pi'\pi'' = 0$. We have constructed a commutative diagram with solid arrows

$$\begin{tikzcd}
C'' \arrow[rr, "\pi''"] \arrow[dr, dashed, "k"] \arrow[ddr, bend right, "h''"'] & & C' \arrow[dd, "\overline{h}'"] \arrow[dr, bend left, "h'"] & \\
& A \arrow[rr, "f"] \arrow[d, "g"] \arrow[dr, phantom, "\lrcorner", very near start] & & B \arrow[d, "g'"] \\
& A' \arrow[r, "\overline{f}'"'] & \mathrm{Im}(f') \arrow[r, hook, "j"'] & B'.
\end{tikzcd}$$

Since the square (3.1) is cartesian, there exists a unique map $k\colon C'' \to A$ such that $gk = h''$ and $fk = h'\pi''$. We now compute

$$h\pi'\pi'' = \pi h'\pi'' = \pi f k = 0.$$

It follows that $\iota\colon \mathrm{Coker}(f) \to \mathrm{Coker}(f')$ is a monomorphism. $\square$

Definition 3.11. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.

(i) F is called *left exact* if it preserves kernels or, equivalently, if for every exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ the induced sequence $0 \to F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ is exact.

(ii) F is called *right exact* if it preserves cokernels or, equivalently, if for every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C \to 0$ in $\mathcal{A}$ the induced sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C) \to 0$ is exact.

(iii) F is called *exact* if it is left exact and right exact.

Proposition 3.12. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories.*

(i) F is left exact if and only if for all short exact sequences $0 \to A \to B \to C \to 0$ the induced sequence $0 \to F(A) \to F(B) \to F(C)$ in $\mathcal{B}$ is exact.

(ii) F is right exact if and only if for all short exact sequences $0 \to A \to B \to C \to 0$ the induced sequence $F(A) \to F(B) \to F(C) \to 0$ in $\mathcal{B}$ is exact.

(iii) F is exact if and only if for every exact sequence $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{A}$ the induced sequence $F(A) \xrightarrow{F(f)} F(B) \xrightarrow{F(g)} F(C)$ in $\mathcal{B}$ is exact.

Proof. We prove (i). The "only if"-direction is trivial. Conversely, assume that F preserves kernels of epimorphisms and in particular monomorphisms. Let now $0 \to K \xrightarrow{i} A \xrightarrow{f} B$ be an exact sequence. We have to show that the induced sequence $0 \to F(K) \xrightarrow{F(i)} F(A) \xrightarrow{F(f)} F(B)$ is exact. By Theorem 3.4(2) we may factor f as a composite $A \xrightarrow{\overline{f}} I \xrightarrow{j} B$, where $\overline{f}$ is an epimorphism and j is a monomorphism. By left exactness, we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & F(K) \arrow[r, "F(i)"] \arrow[d, equal] & F(A) \arrow[r, "F(\overline{f})"] \arrow[d, equal] & F(I) \arrow[d, hook, "F(j)"] \\
0 \arrow[r] & F(K) \arrow[r, "F(i)"'] & F(A) \arrow[r, "F(f)"'] & F(B),
\end{tikzcd}$$

where the top row is exact and $F(j)$ is a monomorphism. In other words, we have $F(K) = \operatorname{Ker}(F(\overline{f}))$. It remains to show that $\operatorname{Ker}(F(\overline{f})) \hookrightarrow F(A)$ is also a kernel for $F(f)$. Let $g\colon X \to F(A)$ be a map such that $F(f) \circ g = 0$. Then $F(j) \circ F(\overline{f}) \circ g = F(f) \circ g = 0$. Since $F(j)$ is a monomorphism, we deduce $F(\overline{f}) \circ g = 0$. Hence, g factors uniquely through $\operatorname{Ker}(F(\overline{f}))$, which shows that $\operatorname{Ker}(F(\overline{f})) = \operatorname{Ker}(F(f))$.

Statement (ii) is dual to (i).

We now prove (iii). The "only if"-direction is obvious. Conversely, suppose that F is exact and let $A \xrightarrow{f} B \xrightarrow{g} C$ be an exact sequence. Split it into an exact sequence $0 \to \operatorname{Ker}(g) \to B \xrightarrow{g} C$ and an epimorphism $A \twoheadrightarrow \operatorname{Ker}(g)$. Since F is exact, we obtain by (i) an exact sequence $0 \to F(\operatorname{Ker}(g)) \to F(B) \xrightarrow{F(g)} F(C)$ and by (ii) an epimorphism $F(A) \twoheadrightarrow F(\operatorname{Ker}(g))$. But this means that $F(A) \to F(B) \to F(C)$ is exact. □

Remark 3.13. Proposition 3.12(iii) shows that $F\colon \mathcal{A} \to \mathcal{B}$ is exact if and only if for every acyclic complex $X^\bullet$ in $\mathcal{A}$ also the complex $F(X^\bullet)$ is acyclic.

Exercise 3.14. Let $\mathcal{A}$ be an abelian category and $A^\bullet$ a complex. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor.

(i) Suppose that F is left exact. Construct a canonical map $\mathrm{H}^0(F(A^\bullet)) \to F(\mathrm{H}^0(A^\bullet))$.

(ii) Suppose that F is right exact. Construct a canonical map $F(\mathrm{H}^0(A^\bullet)) \to \mathrm{H}^0(F(A^\bullet))$.

(iii) Suppose that F is exact. Show that the map in (i) is an isomorphism with inverse given by the map in (ii).

Example 3.15. (a) Every equivalence between abelian categories is exact.

(b) Let $\mathcal{A}$ be an abelian category. For every $A \in \mathcal{A}$ the functors $\operatorname{Hom}_{\mathcal{A}}(A, -): \mathcal{A} \to \mathsf{Ab}$ and $\operatorname{Hom}_{\mathcal{A}}(-, A): \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ are left exact (but generally not right exact).

(c) Let R be a commutative ring. Then for every $M \in \mathsf{Mod}(R)$, the tensor product functor $- \otimes_R M: \mathsf{Mod}(R) \to \mathsf{Mod}(R)$ is right exact (but generally not left exact).

(d) Let $F: \mathcal{A} \rightleftarrows \mathcal{B}: U$ be an adjunction between abelian categories. Then F is right exact and U is left exact. Indeed, as a left adjoint F preserves all colimits (which exist in $\mathcal{A}$), and in particular biproducts, the zero object, and cokernels. Hence, F is additive and right exact by Proposition 3.12(ii). The argument for U is similar.

(e) Let X be a topological space. The global sections functor $\Gamma(X, -)$: $\mathrm{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is left exact but generally not exact. In contrast, the functor $\Gamma(X, -)$: $\mathrm{PSh}(X, \mathsf{Ab}) \to \mathsf{Ab}$ is always exact.

(f) Let k be a field and G a finite group. Denote $\mathsf{Rep}_k(G)$ the category of G-representations on k-vector spaces. The functor $\mathsf{Mod}(k) \to \mathsf{Rep}_k(G)$, which views a k-vector space as a G-representation with the trivial G-action, admits a left adjoint and a right adjoint. The right adjoint is given by the functor of G-invariants:

$$\begin{aligned} \mathsf{Rep}_k(G) &\longrightarrow \mathsf{Mod}(k), \\ V &\longmapsto V^G := \{v \in V \mid gv = v \text{ for all } g \in G\}. \end{aligned}$$

It is therefore left exact; it is exact if and only if the characteristic of k does not divide $|G|$.

The left adjoint is given by the functor of G-coinvariants:

$$\begin{aligned} \mathsf{Rep}_k(G) &\longrightarrow \mathsf{Mod}(k), \\ V &\longmapsto V_G := V/\operatorname{span}_k \{v - gv \mid v \in V, g \in G\}. \end{aligned}$$

It is therefore right exact; it is exact if only if the characteristic of k does not divide $|G|$.

Definition 3.16. Let $\mathcal{A}$ be an abelian category.

(a) An object $I \in \mathcal{A}$ is called *injective* if the following equivalent conditions are satisfied:

(i) The functor $\operatorname{Hom}_{\mathcal{A}}(-, I): \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is exact.

(ii) For every monomorphism $u: A \hookrightarrow B$ and every map $f: A \to I$ there exists a map $g: B \to I$ such that $f = g \circ u$:

$$\begin{tikzcd} & I \\ A \arrow[ur, "f"] \arrow[r, hook, "u"'] & B. \arrow[u, dashed, "\exists g"'] \end{tikzcd}$$

(iii) Every monomorphism $u: I \hookrightarrow B$ splits, *i.e.*, there exists $g: B \to I$ with $g \circ u = \mathrm{id}_I$.

Proof of the equivalences: The equivalence "(i)⇔(ii)" is clear since $\mathrm{Hom}_{\mathcal{A}}(-, I)$ is always left exact, and "(ii)⇒(iii)" is trivial. For "(iii)⇒(ii)", let $u\colon A \hookrightarrow B$ be a monomorphism and $f\colon A \to I$ a map. Consider the pushout diagram

$$\begin{tikzcd} I \arrow[r, hook, "u'"] & I \sqcup_A B \\ A \arrow[u, "f"] \arrow[r, hook, "u"'] \arrow[ur, phantom, "\lrcorner"] & B. \arrow[u, "f'"'] \end{tikzcd}$$

By the addendum to Proposition 3.10(ii), the map u' is monic. Hence there exists a splitting $p\colon I \sqcup_A B \to I$ with $p \circ u' = \mathrm{id}_I$. Then $p \circ f'\colon B \to I$ satisfies $pf'u = pu'f = f$ as desired. ☐

(b) An object $P \in \mathcal{A}$ is called *projective* if the following equivalent conditions are satisfied:

(i) The functor $\mathrm{Hom}_{\mathcal{A}}(P, -)\colon \mathcal{A} \to \mathsf{Ab}$ is exact.

(ii) For every epimorphism $p\colon A \twoheadrightarrow B$ and every map $f\colon P \to B$ there exists a map $g\colon P \to A$ such that $f = p \circ g$:

$$\begin{tikzcd} P \arrow[d, dashed, "\exists g"'] \arrow[dr, "f"] & \\ A \arrow[r, two heads, "p"'] & B. \end{tikzcd}$$

(iii) Every epimorphism $p\colon A \twoheadrightarrow P$ splits, *i.e.*, there exists $g\colon P \to A$ such that $p \circ g = \mathrm{id}_P$.

The following lemma is sometimes useful.

Lemma 3.17. *Let $F\colon \mathcal{A} \rightleftarrows \mathcal{B} : G$ be an adjunction of abelian categories.*

(i) If F is (left) exact, then G preserves injective objects. The converse holds if $\mathcal{B}$ has enough injectives.

(ii) If G is (right) exact, then F preserves projective objects. The converse holds if $\mathcal{A}$ has enough projectives.

Proof. Let us only prove (i), because (ii) is dual. Let $I \in \mathcal{B}$ be injective. The adjunction then gives a natural isomorphism

$$\mathrm{Hom}_{\mathcal{A}}(-, G(I)) \cong \mathrm{Hom}_{\mathcal{B}}(-, I) \circ F^{\mathrm{op}}$$

of functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$. Since both functors on the right are exact, so is the composition, and hence $G(I)$ is injective.

For the converse, suppose that $\mathcal{B}$ has enough injectives and G preserves injectives. Let $u\colon A \hookrightarrow A'$ be a monomorphism in $\mathcal{A}$. We need to show that $F(u)$ is a monomorphism. Choose a monomorphism $f\colon F(A) \hookrightarrow I$. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{B}}(F(A'), I) \arrow[r, "F(u)^*"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathcal{B}}(F(A), I) \arrow[d, "\sim"] \\ \mathrm{Hom}_{\mathcal{A}}(A', G(I)) \arrow[r, two heads, "u^*"'] & \mathrm{Hom}_{\mathcal{A}}(A, G(I)), \end{tikzcd}$$

where the bottom map is an epimorphism because $G(I)$ is injective. It follows that the top map is an epimorphism. Hence, there exists $g\colon F(A') \to I$ such that $g \circ F(u) = f$. As f is a monomorphism, it follows that $F(u)$ is a monomorphism as well. $\square$

Lemma 3.18. *Let $\mathcal{A}$ be an abelian category.*

(i) Let $(I_j)_{j\in J}$ be a family of objects such that the product $I = \prod_{j\in J} I_j$ exists. Then I is injective if and only if, for all $j \in J$, the object I_j is injective.

(ii) Let $(P_j)_{j\in J}$ be a family of objects such that the direct sum $P = \bigoplus_{j\in J} P_j$ exists. Then P is projective if and only if, for all $j \in J$, the object P_j is injective.

Proof. We only prove (i), because (ii) is completely dual. We have an isomorphism $\mathrm{Hom}_{\mathcal{A}}(-, I) \cong \prod_{j\in J} \mathrm{Hom}_{\mathcal{A}}(-, I_j)$ of functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$. As the formation of infinite products in Ab is exact, we conclude that, if each I_j is injective, then so is I. Conversely, if I is injective, then the exactness of $\prod_{j\in J} \mathrm{Hom}_{\mathcal{A}}(-, I_j)$ implies that each $\mathrm{Hom}_{\mathcal{A}}(-, I_j)$ is exact. Hence each I_j is injective. $\square$

Chapter 2

Triangulated Categories

§4. The axioms of a triangulated category

We begin by stating the definition of triangulated categories and drawing some consequences of the axioms.

Definition 4.1. A *category with translation* is a pair $(\mathcal{C}, T)$ consisting of a category $\mathcal{C}$ and a self-equivalence $T\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$, called a *translation functor*. We will often write $\mathcal{C}$ instead of $(\mathcal{C}, T)$ when no confusion arises.

A *triangle* in $(\mathcal{C}, T)$ is a sequence

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$$

in $\mathcal{C}$. A triangle is sometimes denoted by (X, Y, Z, u, v, w) or simply by (u, v, w).

A *morphism of triangles* $(x, y, z)\colon (X, Y, Z, u, v, w) \to (X', Y', Z', u', v', w')$ consists of three morphisms $x\colon X \to X'$, $y\colon Y \to Y'$, $z\colon Z \to Z'$ in $\mathcal{C}$ such that the diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[r, "v"] \arrow[d, "y"] & Z \arrow[r, "w"] \arrow[d, "z"] & T(X) \arrow[d, "T(x)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X')
\end{tikzcd}$$

is commutative.

Definition 4.2. A *triangulated category* consists is a triple $(\mathcal{C}, T, \mathcal{DT})$, where $(\mathcal{C}, T)$ is an additive category with translation and $\mathcal{DT}$ is a class of triangles in $(\mathcal{C}, T)$, which we call *distinguished triangles*, subject to the following conditions:

(T1) (Verdier's axiom TR1)

- (a) $\mathcal{DT}$ is closed under isomorphisms of triangles.
- (b) The triangle $X \xrightarrow{\mathrm{id}_X} X \to 0 \to T(X)$ is distinguished.
- (c) Every morphism $u\colon X \to Y$ in $\mathcal{C}$ sits in a distinguished triangle $X \xrightarrow{u} Y \to Z \to T(X)$.

(T2) (Verdier's axiom TR2: first half)

If $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is a distinguished triangle, then so is $Y \xrightarrow{v} Z \xrightarrow{w} T(X) \xrightarrow{-T(u)} T(Y)$.[1]

(T3) (Verdier's axiom TR4)

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathcal{C}$, put $h \coloneqq g \circ f\colon X \to Y$, and suppose we have distinguished triangles

$$\begin{aligned} X &\xrightarrow{f} Y \xrightarrow{f'} Y/X \xrightarrow{f''} T(X), \\ Y &\xrightarrow{g} Z \xrightarrow{g'} Z/Y \xrightarrow{g''} T(Y), \\ X &\xrightarrow{h} Z \xrightarrow{h'} Z/X \xrightarrow{h''} T(X). \end{aligned}$$

Then there exist morphisms $u\colon Y/X \to Z/X$ and $v\colon Z/X \to Z/Y$ in $\mathcal{C}$ such that

$$Y/X \xrightarrow{u} Z/X \xrightarrow{v} Z/Y \xrightarrow{T(f')g''} T(Y/X)$$

is a distinguished triangle and such that the following "braid diagram" commutes:

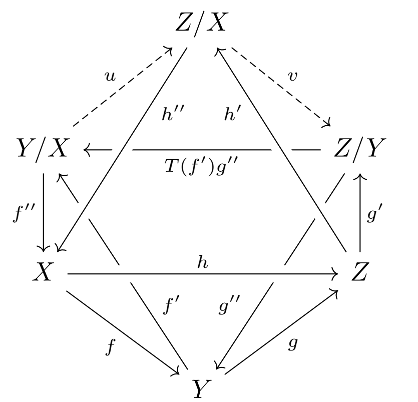
$$\begin{tikzcd}
& & Y/X \arrow[rr, bend left, "f''"] \arrow[dr, dashed, "u"] & & T(X) \arrow[dr, "T(f)"] & & \\
& Y \arrow[ur, "f'"] \arrow[dr, "g"] & & Z/X \arrow[ur, "h''"] \arrow[dr, dashed, "v"] & & T(Y) \arrow[dr, "T(f')"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Z \arrow[ur, "h'"] \arrow[rr, bend right, "g'"'] & & Z/Y \arrow[ur, "g''"] \arrow[rr, bend right, "T(f')g''"'] & & T(Y/X).
\end{tikzcd}$$

We will always denote a triangulated category by $\mathcal{C}$ or $(\mathcal{C}, T)$ and leave the class $\mathcal{DT}$ of distinguished triangles implicit (although they are additional data!).

Remark 4.3. (a) The axiom (T3) is usually referred to as the *octahedral axiom*, because the

[1] For the theory of triangulated categories, it would probably be equally good to replace $-T(u)$ with $T(u)$. But then there would probably not be many interesting triangulated categories: The sign is necessary to ensure that the homotopy category $\mathsf{K}(\mathcal{A})$ is triangulated (Theorem 4.12).

displayed diagram can be arranged in the form of an octahedron:



Note that the formulation in [Har66] is insufficient as it fails to require that the squares in the octahedron commute.

(b) The original axioms of Verdier [Ver96, Ch. II, Définition 1.1.1] are actually redundant as has been observed by May [May05], and we will thus follow May's approach and deduce Verdier's axiom TR3 from the other ones.

(c) If (u, v, w) is a distinguished triangle, then so are $(u, -v, -w)$, $(-u, v, -w)$ and $(-u, -v, w)$ by (T1), because they are isomorphic to (u, v, w). For example, we have an isomorphism

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "\mathrm{id}_X"'] & Y \arrow[r, "v"] \arrow[d, "\mathrm{id}_Y"] & W \arrow[r, "w"] \arrow[d, "-\mathrm{id}_W"] & T(X) \arrow[d, "T(\mathrm{id}_X)"] \\ X \arrow[r, "u"'] & Y \arrow[r, "-v"'] & W \arrow[r, "-w"'] & T(X). \end{tikzcd}$$

Before we can give an example of a triangulated category we need to develop a little bit of the theory first. Since our definition of triangulated category does not involve Verdier's axiom TR3 and only half of TR2, we first show that these are actually consequences of the axioms (T1)–(T3).

Proposition 4.4 (Verdier's axiom TR3)**.** *Let $(\mathcal{C}, T)$ be a triangulated category. Consider a diagram with solid arrows*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[r, "v"] \arrow[d, "y"'] & Z \arrow[r, "w"] \arrow[d, dashed, "z"] & T(X) \arrow[d, "T(x)"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X'), \end{tikzcd}$$

where the left square commutes and the rows are distinguished triangles. Then there exists $z\colon Z \to Z'$ in $\mathcal{C}$ such that the whole diagram commutes, that is, $z \circ v = v' \circ y$ and $T(x) \circ w = w' \circ z$.

Note that the morphism $z\colon Z \to Z'$ in the proposition need not be unique; sufficent conditions for uniqueness will be provided in Lemma 5.1.

Proof. By (T1) we may embed x, y and $y \circ u = u' \circ x$ in distinguished triangles

$$\begin{aligned} X \xrightarrow{x} X' \xrightarrow{x'} X'' \xrightarrow{x''} T(X), \\ Y \xrightarrow{y} Y' \xrightarrow{y'} Y'' \xrightarrow{y''} T(Y), \\ X \xrightarrow{yu} Y' \xrightarrow{p} W \xrightarrow{q} T(X). \end{aligned}$$

We apply (T3) to the pairs of morphisms (y, u) and (u', x), respectively. We thus obtain new distinguished triangles

$$\begin{aligned} Z \xrightarrow{s} W \xrightarrow{t} Y'' \xrightarrow{T(v)y''} T(Z), \\ X'' \xrightarrow{s'} W \xrightarrow{t'} Z' \xrightarrow{T(x')w'} T(X'') \end{aligned}$$

such that the following diagrams commute:

$$\begin{tikzcd}
 & & Z \arrow[rr, bend left, "w"] \arrow[dr, dashed, "s"] & & T(X) \arrow[dr, "T(u)"] & & \\
 & Y \arrow[ur, "v"] \arrow[dr, "y"] & & W \arrow[ur, "q"] \arrow[dr, dashed, "t"] & & T(Y) \arrow[dr, "T(v)"] & \\
X \arrow[ur, "u"] \arrow[rr, bend right, "yu"'] & & Y' \arrow[ur, "p"] \arrow[rr, bend right, "y'"'] & & Y'' \arrow[ur, "y''"] \arrow[rr, bend right, "T(v)y''"'] & & T(Z)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & X'' \arrow[rr, bend left, "x''"] \arrow[dr, dashed, "s'"] & & T(X) \arrow[dr, "T(x)"] & & \\
 & X' \arrow[ur, "x'"] \arrow[dr, "u'"] & & W \arrow[ur, "q"] \arrow[dr, dashed, "t'"] & & T(X') \arrow[dr, "T(x')"] & \\
X \arrow[ur, "x"] \arrow[rr, bend right, "u'x"'] & & Y' \arrow[ur, "p"] \arrow[rr, bend right, "v'"'] & & Z' \arrow[ur, "w'"] \arrow[rr, bend right, "T(x')w'"'] & & T(X'').
\end{tikzcd}$$

We define $z := t' \circ s \colon Z \to Z'$ and compute

$$\begin{aligned} zv = t'sv = t'py = v'y, \\ w'z = w't's = T(x)qs = T(x)w, \end{aligned}$$

which proves the assertion. $\square$

Definition 4.5. Let $(\mathcal{C}, T)$ be a triangulated category and $\mathcal{A}$ an abelian category. An additive functor $H\colon \mathcal{C} \to \mathcal{A}$ is called *cohomological* if every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ in $\mathcal{C}$ induces a long exact sequence

$$\cdots \to H(T^i(X)) \xrightarrow{T^i(u)} H(T^i(Y)) \xrightarrow{T^i(v)} H(T^i(Z)) \xrightarrow{T^i(w)} H(T^{i+1}(X)) \longrightarrow \cdots$$

Example 4.6. Let $\mathcal{A}$ be an abelian category. Then the functor $\mathrm{H}^0\colon \mathsf{K}(\mathcal{A}) \to \mathcal{A}$ is cohomological. Indeed, let $f\colon X \to Y$ be a morphism of complexes and consider the associated triangle $X \xrightarrow{f} Y \xrightarrow{i_Y} \mathrm{Mc}(f) \xrightarrow{p_X} X[1]$. We have to show that for all $i \in \mathbb{Z}$ the sequence

$$\cdots \to \mathrm{H}^0(X[i]) \to \mathrm{H}^0(Y[i]) \to \mathrm{H}^0(\mathrm{Mc}(f)[i]) \to \mathrm{H}^0(X[i+1]) \to \cdots \tag{4.1}$$

is exact. Note that we have a short exact sequence $0 \to Y \to \mathrm{Mc}(f) \to X[1] \to 0$ in $\mathsf{C}(\mathcal{A})$. Hence, we obtain a long exact sequence in cohomology

$$\cdots \to \mathrm{H}^i(Y) \to \mathrm{H}^i(\mathrm{Mc}(f)) \to \mathrm{H}^i(X[1]) \xrightarrow{\partial^i} \mathrm{H}^{i+1}(Y) \to \cdots .$$

We claim that this sequence is the same as (4.1), which then proves the claim. We clearly have $\mathrm{H}^i(C[j]) = \mathrm{H}^{i+j}(C)$ for all $i, j \in \mathbb{Z}$ and all $C \in \mathsf{K}(\mathcal{A})$. It remains to show that $\partial^i = \mathrm{H}^{i+1}(f)$ as maps $\mathrm{H}^{i+1}(X) = \mathrm{H}^i(X[1]) \to \mathrm{H}^{i+1}(Y)$. Recall the definition of ∂^i: We have a commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & \mathrm{H}^i(Y) \arrow[r] \arrow[d, hook] & \mathrm{H}^i(\mathrm{Mc}(f)) \arrow[r] \arrow[d, hook] & \mathrm{H}^i(X[1]) \arrow[d, hook] \arrow[dddll, dotted, rounded corners, "\partial^i"] & \\
& Y^i/\operatorname{Im}(d_Y^{i-1}) \arrow[r] \arrow[d, "d_Y^i"] & (X^{i+1} \oplus Y^i)/\operatorname{Im}(d_{\mathrm{Mc}(f)}^{i-1}) \arrow[r] \arrow[d, "d_{\mathrm{Mc}(f)}^i"] & X^{i+1}/\operatorname{Im}(d_X^i) \arrow[r] \arrow[d, "d_X^{i+1}"] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d_Y^{i+1}) \arrow[r] \arrow[d, two heads] & \operatorname{Ker}(d_{\mathrm{Mc}(f)}^{i+1}) \arrow[r] \arrow[d, two heads] & \operatorname{Ker}(d_X^{i+2}) \arrow[d, two heads] & \\
& \mathrm{H}^{i+1}(Y) \arrow[r] & \mathrm{H}^{i+1}(\mathrm{Mc}(f)) \arrow[r] & \mathrm{H}^{i+1}(X[1]) \arrow[r] & \cdots
\end{tikzcd}$$

Consider $x \in \operatorname{Ker}(d_X^{i+1})$ with image $\overline{x} \in \mathrm{H}^{i+1}(X)$. We may lift x to $(x, 0) \in \mathrm{Mc}(f)^i$, and then $\partial^i(\overline{x})$ is the image of $d_{\mathrm{Mc}(f)}^i(x, 0) = (0, f^{i+1}(x))$ in $\mathrm{H}^{i+1}(Y)$. This shows $\partial^i = \mathrm{H}^{i+1}(f)$.

Proposition 4.7. *Let $(\mathcal{C}, T)$ be a triangulated category.*

(i) The composition of any two consecutive morphisms in a distinguished triangle is zero.

(ii) For each $C \in \mathcal{C}$ the functors $\operatorname{Hom}_{\mathcal{C}}(C, -)\colon \mathcal{C} \to \mathsf{Ab}$ and $\operatorname{Hom}_{\mathcal{C}}(-, C)\colon \mathcal{C} \to \mathsf{Ab}^{\mathrm{op}}$ are cohomological.

(iii) Consider a morphism of triangles

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "x"'] & Y \arrow[r] \arrow[d, "y"] & Z \arrow[r] \arrow[d, "z"] & T(X) \arrow[d, "T(x)"] \\
X' \arrow[r] & Y' \arrow[r] & Z' \arrow[r] & T(X').
\end{tikzcd}$$

If any two of x, y, z are isomorphisms, then so is the third.

Proof. We first prove (i), so let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ be a distinguished triangle in $\mathcal{C}$. By (T2) also the triangles $(v, w, -T(u))$ and $(w, -T(u), -T(v))$ are distinguished. It therefore suffices to show $vu = 0$. To this end, we apply Proposition 4.4 to the distinguished triangles $(v, w, -T(u))$ and $(\mathrm{id}_Z, 0, 0)$ (which is distinguished by (T1)) and obtain a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "v"] \arrow[d, "v"'] & Z \arrow[r, "w"] \arrow[d, "\mathrm{id}_Z"] & T(X) \arrow[r, "-T(u)"] \arrow[d, dashed, "0"] & T(Y) \arrow[d, "T(v)"] \\
Z \arrow[r, "\mathrm{id}_Z"'] & Z \arrow[r] & 0 \arrow[r] & T(Z).
\end{tikzcd}$$

We deduce $T(vu) = -T(v) \circ (-T(u)) = 0$. As T is an equivalence, this shows $vu = 0$.

We next prove (ii). We only show that $\mathrm{Hom}_{\mathcal{C}}(C, -)$ is cohomological, because the argument for $\mathrm{Hom}_{\mathcal{C}}(-, C)$ is completely analogous. Let (u, v, w) be a distinguished triangle as above. It suffices to show that the sequence

$$\mathrm{Hom}_{\mathcal{C}}(C, X) \xrightarrow{u_*} \mathrm{Hom}_{\mathcal{C}}(C, Y) \xrightarrow{v_*} \mathrm{Hom}_{\mathcal{C}}(C, Z)$$

is exact, because exactness of the long sequence at every other place follows by repeated applications of (T2) and the fact that T is an equivalence. By (i) we know $v_* \circ u_* = 0$, which shows $\mathrm{Im}(u_*) \subseteq \mathrm{Ker}(v_*)$. For the reverse inclusion, let $f\colon C \to Y$ be a morphism in $\mathcal{C}$ such that $v \circ f = 0$. We need to find $g\colon C \to X$ such that $f = u \circ g$. By (T1) the triangle $(\mathrm{id}_C, 0, 0)$ is distinguished. Hence, by (T2) we obtain distinguished triangles $(0, 0, -T(\mathrm{id}_C))$ and $(v, w, -T(u))$ and then Proposition 4.4 provides a commutative diagram

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "f"'] & 0 \arrow[r] \arrow[d] & T(C) \arrow[r, "-T(\mathrm{id}_C)"] \arrow[d, dashed, "g'"] & T(C) \arrow[d, "T(f)"] \\
Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & T(X) \arrow[r, "-T(u)"'] & T(Y).
\end{tikzcd}$$

For the unique map $g\colon C \to X$ with $T(g) = g'$, we obtain

$$u \circ g = -T^{-1}\big(-T(u) \circ g'\big) = -T^{-1}\big(T(f) \circ (-T(\mathrm{id}_C))\big) = f,$$

which shows $\mathrm{Ker}(v_*) \subseteq \mathrm{Im}(u_*)$ as desired.

Finally, we prove (iii). Consider the commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "x"'] & Y \arrow[r, "v"] \arrow[d, "y"] & Z \arrow[r, "w"] \arrow[d, "z"] & T(X) \arrow[d, "T(x)"] \\
X' \arrow[r, "u'"'] & Y' \arrow[r, "v'"'] & Z' \arrow[r, "w'"'] & T(X'),
\end{tikzcd}$$

where the rows are distinguished triangles. We only show that if x and y are isomorphisms, then so is z; the other cases are similar. By (ii) the functor $\mathrm{Hom}_{\mathcal{C}}(C, -)$ is cohomological for all $C \in \mathcal{C}$. We thus obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(C, X) \arrow[r] \arrow[d, "x_*"] & \mathrm{Hom}(C, Y) \arrow[r] \arrow[d, "y_*"] & \mathrm{Hom}(C, Z) \arrow[r] \arrow[d, "z_*"] & \mathrm{Hom}(C, T(X)) \arrow[r] \arrow[d, "T(x)_*"] & \mathrm{Hom}(C, T(Y)) \arrow[d, "T(y)_*"] \\
\mathrm{Hom}(C, X') \arrow[r] & \mathrm{Hom}(C, Y') \arrow[r] & \mathrm{Hom}(C, C) \arrow[r] & \mathrm{Hom}(C, T(X')) \arrow[r] & \mathrm{Hom}(C, T(Y'))
\end{tikzcd}$$

with exact rows and where all vertical arrows (except z_*) are known to be isomorphisms. By the five lemma, it follows that z_* is an isomorphism. By the Yoneda lemma, we deduce that z is an isomorphism. □

Remark. Note that the proof of Proposition 4.7 depends only on the axioms (T1), (T2) and Proposition 4.4, but not on the octahedral axiom.

Exercise 4.8. Let $\mathcal{A}$ be an additive category. Show that a morphism $f\colon X \to Y$ of complexes is a homotopy equivalence if and only if $\mathrm{Mc}(f)$ is contractible.

Exercise 4.9. Let $(\mathcal{C}, T, \mathcal{DT}_1)$ and $(\mathcal{C}, T, \mathcal{DT}_2)$ be two triangulated structures on a category with translation $(\mathcal{C}, T)$ such that $\mathcal{DT}_1$ is contained in $\mathcal{DT}_2$. Show that $\mathcal{DT}_1 = \mathcal{DT}_2$.

Proposition 4.10 (Verdier's axiom TR2: second half)**.** *Let $(\mathcal{C}, T)$ be a triangulated category and $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ a triangle. Then (u, v, w) is distinguished if and only if the shifted triangle $(v, w, -T(u))$ is distinguished.*

Proof. The "only if" direction is (T2). We thus need to prove the converse direction. By (T1) there exists a distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v'} Z' \xrightarrow{w'} T(X)$. Applying (T2) repeatedly, we find that the triangles $(-T(u), -T(v'), -T(w'))$ and $(-T(u), -T(v), -T(w))$ are distinguished, and by Proposition 4.4 we obtain a commutative diagram

$$\begin{tikzcd}
T(X) \arrow[r, "-T(u)"] \arrow[d, equal] & T(Y) \arrow[r, "-T(v')"] \arrow[d, equal] & T(Z') \arrow[r, "-T(w')"] \arrow[d, dashed, "f'"] & T^2(X) \arrow[d, equal] \\
T(X) \arrow[r, "-T(u)"'] & T(Y) \arrow[r, "-T(v)"'] & T(Z) \arrow[r, "-T(w)"'] & T^2(X),
\end{tikzcd}$$

where $f'\colon T(Z') \xrightarrow{\sim} T(Z)$ is an isomorphism by Proposition 4.7(iii). Since T is an equivalence, there exists a unique isomorphism $f\colon Z' \xrightarrow{\sim} Z$ such that $T(f) = f'$. We obtain a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v'"] \arrow[d, equal] & Z' \arrow[r, "w'"] \arrow[d, "f"] & T(X) \arrow[d, equal] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & T(X),
\end{tikzcd}$$

where the top triangle is distinguished. By (T1) also the bottom triangle is distinguished. □

Exercise 4.11. Let $\mathcal{C}$ be an additive category, let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in $\mathsf{C}(\mathcal{C})$ such that $g \circ f$ is homotopic to zero. Show that g factors through $\mathrm{Mc}(f)$.

We have finally shown that our definition of "triangulated category" coincides with the classical definition. We now have everything we need to give a first example of a triangulated category.

Theorem 4.12. *Let $\mathcal{A}$ be an additive category. Let $\mathcal{DT}$ be the class of triangles in $(\mathsf{K}(\mathcal{A}), [1])$ which are isomorphic to a triangle of the form*

$$X \xrightarrow{f} Y \xrightarrow{\iota} \mathrm{Mc}(f) \xrightarrow{\pi} X[1],$$

where ι and π are the canonical maps. Then $(\mathsf{K}(\mathcal{A}), [1], \mathcal{DT})$ is a triangulated category.

The categories $\mathsf{K}^+(\mathcal{A})$, $\mathsf{K}^-(\mathcal{A})$ and $\mathsf{K}^b(\mathcal{A})$ are triangulated subcategories of $\mathsf{K}(\mathcal{A})$.

Proof. The last statement is obvious, so we only need to prove that $\mathsf{K}(\mathcal{A})$ is triangulated.

Note that $\mathrm{Mc}(\mathrm{id}_X) = 0$ in $\mathsf{K}(\mathcal{A})$ by Proposition 2.13(i). Hence (T1) is trivially satisfied.

We now prove (T2). Note that the condition in (T2) is invariant under isomorphisms of triangles. Consider the distinguished triangle $X \xrightarrow{f} Y \xrightarrow{\iota} \mathrm{Mc}(f) \xrightarrow{\pi} X[1]$, where $\iota = \left(\begin{smallmatrix} 0 \\ \mathrm{id}_Y \end{smallmatrix}\right)$ and $\pi = (\mathrm{id}_X, 0)$. Note that $\mathrm{Mc}(\iota)^i = Y^{i+1} \oplus X^{i+1} \oplus Y^i$ with differential $d = \left(\begin{smallmatrix} -d_Y & 0 & 0 \\ 0 & -d_X & 0 \\ \mathrm{id}_Y & f & d_Y \end{smallmatrix}\right)$. We construct an isomorphism of triangles

$$\begin{tikzcd} Y \arrow[r, "\iota"] \arrow[d, "\mathrm{id}"'] & \mathrm{Mc}(f) \arrow[r, "j"] \arrow[d, "\mathrm{id}"] & \mathrm{Mc}(\iota) \arrow[r, "p"] \arrow[d, "g"] & Y[1] \arrow[d, "\mathrm{id}"] \\ Y \arrow[r, "\iota"'] & \mathrm{Mc}(f) \arrow[r, "\pi"'] & X[1] \arrow[r, "-f[1]"'] & Y[1] \end{tikzcd} \tag{4.2}$$

in $\mathsf{K}(\mathcal{A})$ via

$$\begin{aligned} g^i = \begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix} : \mathrm{Mc}(\iota)^i = Y^{i+1} \oplus X^{i+1} \oplus Y^i &\longrightarrow X^{i+1} \\ (y, x, y') &\longmapsto x \end{aligned}$$

for all $i \in \mathbb{Z}$. From the definition it is clear that g is a morphism of complexes and that $g \circ j = \pi \circ \mathrm{id}_{\mathrm{Mc}(f)}$ already holds in $\mathsf{C}(\mathcal{A})$. It remains to prove $-f[1] \circ g = \mathrm{id}_{Y[1]} \circ p$ and that g is a homotopy equivalence.

We first show that $p + f[1] \circ g\colon \mathrm{Mc}(\iota) \to Y[1]$ is null homotopic. To this end, we define

$$\begin{aligned} s^i = \begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix} : \mathrm{Mc}(\iota)^i = Y^{i+1} \oplus X^{i+1} \oplus Y^i &\longrightarrow Y^i = Y[1]^{i-1}, \\ (y, x, y') &\longmapsto y'. \end{aligned}$$

Recalling $d_{Y[1]} = -d_Y$, we then compute

$$\begin{aligned} s^{i+1} d^i_{\mathrm{Mc}(\iota)} + d^{i-1}_{Y[1]} s^i &= \begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ \mathrm{id} & f & d \end{pmatrix} + (-d) \circ \begin{pmatrix} 0 & 0 & \mathrm{id} \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & f & d \end{pmatrix} + \begin{pmatrix} 0 & 0 & -d \end{pmatrix} = \begin{pmatrix} \mathrm{id} & f & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & 0 & 0 \end{pmatrix} + f[1] \circ \begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix} = p + f[1] \circ g. \end{aligned}$$

Hence, the diagram (4.2) is commutative in $\mathsf{K}(\mathcal{A})$.

It remains to show that g is a homotopy equivalence. Define a map $h\colon X[1] \to \mathrm{Mc}(\iota)$ via

$$\begin{aligned} h^i = \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} : X[1]^i = X^{i+1} &\longrightarrow Y^{i+1} \oplus X^{i+1} \oplus Y^i = \mathrm{Mc}(\iota)^i, \\ x &\longmapsto (-f(x), x, 0) \end{aligned}$$

for $i \in \mathbb{Z}$. Again, keeping in mind that $d_{X[1]} = -d_X$, we compute

$$\begin{aligned} h^{i+1}d^i_{X[1]} - d^i_{\mathrm{Mc}(\iota)}h^i &= \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} \circ (-d) - \begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ \mathrm{id} & f & d \end{pmatrix} \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} fd \\ -d \\ 0 \end{pmatrix} - \begin{pmatrix} df \\ -d \\ -f+f \end{pmatrix} = 0. \end{aligned}$$

We deduce that h is a morphism of complexes because f is. Moreover, we have $g \circ h = \mathrm{id}_{X[1]}$ already in $\mathsf{C}(\mathcal{A})$. It thus remains to show that $\mathrm{id}_{\mathrm{Mc}(\iota)} - h \circ g$ is null homotopic. To this end, we define

$$\begin{aligned} t^i = \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathrm{Mc}(\iota)^i = Y^{i+1} \oplus X^{i+1} \oplus Y^i &\longrightarrow Y^i \oplus X^i \oplus Y^{i-1} = \mathrm{Mc}(\iota)^{i-1}, \\ (y, x, y') &\longmapsto (y', 0, 0). \end{aligned}$$

Now, compute

$$\begin{aligned} t^{i+1}d^i_{\mathrm{Mc}(\iota)} + d^{i-1}_{\mathrm{Mc}(\iota)}t^i &= \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ \mathrm{id} & f & d \end{pmatrix} + \begin{pmatrix} -d & 0 & 0 \\ 0 & -d & 0 \\ \mathrm{id} & f & d \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{id} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} \mathrm{id} & f & d \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -d \\ 0 & 0 & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & \mathrm{id} \end{pmatrix} - \begin{pmatrix} 0 & -f & 0 \\ 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &= \mathrm{id}_{\mathrm{Mc}(\iota)} - \begin{pmatrix} -f \\ \mathrm{id} \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} & 0 \end{pmatrix} \\ &= \mathrm{id}_{\mathrm{Mc}(\iota)} - h \circ g. \end{aligned}$$

We deduce that g is a homotopy equivalence with homotopy inverse h. This finishes the verification of axiom (T2).

Finally, we need to prove the octahedral axiom (T3). We first handle a special case: Consider distinguished triangles

$$\begin{aligned} X \xrightarrow{f} Y &\xrightarrow{\iota_f} \mathrm{Mc}(f) \xrightarrow{\pi_f} X[1], \\ Y \xrightarrow{g} Z &\xrightarrow{\iota_g} \mathrm{Mc}(g) \xrightarrow{\pi_g} Y[1], \\ X \xrightarrow{gf} Z &\xrightarrow{\iota_{gf}} \mathrm{Mc}(gf) \xrightarrow{\pi_{gf}} X[1] \end{aligned}$$

in $\mathsf{K}(\mathcal{A})$. We need to construct a distinguished triangle

$$\mathrm{Mc}(f) \xrightarrow{u} \mathrm{Mc}(gf) \xrightarrow{v} \mathrm{Mc}(g) \xrightarrow{\iota_f[1]\pi_g} \mathrm{Mc}(f)[1]$$

such that the diagram

$$
\begin{tikzcd}
 & & \mathrm{Mc}(f) \arrow[rr, bend left, "\pi_f"] \arrow[dr, dashed, "u"] & & X[1] \arrow[dr, "{f[1]}"] & & \\
 & Y \arrow[ur, "\iota_f"] \arrow[dr, "g"] & & \mathrm{Mc}(gf) \arrow[ur, "\pi_{gf}"] \arrow[dr, dashed, "v"] & & Y[1] \arrow[dr, "{\iota_f[1]}"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "gf"'] & & Z \arrow[ur, "\iota_{gf}"] \arrow[rr, bend right, "\iota_g"'] & & \mathrm{Mc}(g) \arrow[ur, "\pi_g"] \arrow[rr, bend right, "{\iota_f[1]\pi_g}"'] & & \mathrm{Mc}(f)[1]
\end{tikzcd}
\tag{4.3}
$$

commutes in $\mathsf{K}(\mathcal{A})$.

Step 1: We define morphisms $u\colon \mathrm{Mc}(f) \to \mathrm{Mc}(gf)$ and $v\colon \mathrm{Mc}(gf) \to \mathrm{Mc}(g)$ via

$$
\begin{aligned}
u^i := \begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix}\colon \mathrm{Mc}(f)^i = X^{i+1} \oplus Y^i &\longrightarrow X^{i+1} \oplus Z^i = \mathrm{Mc}(gf)^i, \\
(x, y) &\longmapsto (x, g(y)), \\
v^i := \begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix}\colon \mathrm{Mc}(gf)^i = X^{i+1} \oplus Z^i &\longrightarrow Y^{i+1} \oplus Z^i = \mathrm{Mc}(g)^i, \\
(x, z) &\longmapsto (f(x), z).
\end{aligned}
$$

The computations

$$
\begin{aligned}
ud_{\mathrm{Mc}(f)} - d_{\mathrm{Mc}(gf)}u &= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix}\begin{pmatrix} -d & 0 \\ f & d \end{pmatrix} - \begin{pmatrix} -d & 0 \\ gf & d \end{pmatrix}\begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix} \\
&= \begin{pmatrix} -d & 0 \\ gf & gd \end{pmatrix} - \begin{pmatrix} -d & 0 \\ gf & dg \end{pmatrix} = \begin{pmatrix} -d + d & 0 \\ gf - gf & gd - dg \end{pmatrix} = 0, \\
vd_{\mathrm{Mc}(gf)} - d_{\mathrm{Mc}(g)}v &= \begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix}\begin{pmatrix} -d & 0 \\ gf & d \end{pmatrix} - \begin{pmatrix} -d & 0 \\ g & d \end{pmatrix}\begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix} \\
&= \begin{pmatrix} -fd & 0 \\ gf & d \end{pmatrix} - \begin{pmatrix} -df & 0 \\ gf & d \end{pmatrix} = \begin{pmatrix} -fd + df & 0 \\ gf - gf & d - d \end{pmatrix} = 0
\end{aligned}
$$

show that u and v are indeed morphisms of complexes.

Step 2: We construct a homotopy equivalence $\varphi\colon \mathrm{Mc}(u) \to \mathrm{Mc}(g)$ such that the diagram

$$
\begin{tikzcd}
\mathrm{Mc}(f) \arrow[r, "u"] \arrow[d, equal] & \mathrm{Mc}(gf) \arrow[r, "\iota_u"] \arrow[d, equal] & \mathrm{Mc}(u) \arrow[r, "\pi_u"] \arrow[d, "\varphi"] & \mathrm{Mc}(f)[1] \arrow[d, equal] \\
\mathrm{Mc}(f) \arrow[r, "u"'] & \mathrm{Mc}(gf) \arrow[r, "v"'] & \mathrm{Mc}(g) \arrow[r, "{\iota_f[1]\pi_g}"'] & \mathrm{Mc}(f)[1]
\end{tikzcd}
\tag{4.4}
$$

commutes. To this end, we put

$$
\begin{aligned}
\varphi^i := \begin{pmatrix} 0 & \mathrm{id} & f & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix}\colon \mathrm{Mc}(u)^i = X^{i+2} \oplus Y^{i+1} \oplus X^{i+1} \oplus Z^i &\longrightarrow Y^{i+1} \oplus Z^i = \mathrm{Mc}(g)^i, \\
(x, y, x', z) &\longmapsto (y + f(x'), z)
\end{aligned}
$$

and we claim that the homotopy inverse $\psi\colon \operatorname{Mc}(g) \to \operatorname{Mc}(u)$ is given by

$$\psi^i := \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix}\colon \operatorname{Mc}(g)^i = Y^{i+1} \oplus Z^i \longrightarrow X^{i+2} \oplus Y^{i+1} \oplus X^{i+1} \oplus Z^i = \operatorname{Mc}(u)^i,$$
$$(y, z) \longmapsto (0, y, 0, z).$$

The computations

$$\begin{aligned}
\varphi d_{\operatorname{Mc}(u)} - d_{\operatorname{Mc}(g)}\varphi &= \begin{pmatrix} 0 & \mathrm{id} & f & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 \\ -f & -d & 0 & 0 \\ \mathrm{id} & 0 & -d & 0 \\ 0 & g & gf & d \end{pmatrix} - \begin{pmatrix} -d & 0 \\ g & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} & f & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \\
&= \begin{pmatrix} -f+f & -d & -fd & 0 \\ 0 & 0 & 0 & d \end{pmatrix} - \begin{pmatrix} 0 & -d & -df & 0 \\ 0 & g & gf & d \end{pmatrix} = 0, \\
\psi d_{\operatorname{Mc}(g)} - d_{\operatorname{Mc}(u)}\psi &= \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} -d & 0 \\ g & d \end{pmatrix} - \begin{pmatrix} d & 0 & 0 & 0 \\ -f & -d & 0 & 0 \\ \mathrm{id} & 0 & -d & 0 \\ 0 & g & gf & d \end{pmatrix} \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ -d & 0 \\ 0 & 0 \\ g & d \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -d & 0 \\ 0 & 0 \\ g & d \end{pmatrix} = 0
\end{aligned}$$

show that φ and ψ are indeed morphisms of complexes. Note that we have $\varphi \circ \psi = \mathrm{id}_{\operatorname{Mc}(g)}$ already in $\mathsf{C}(\mathcal{A})$. We next show that $\mathrm{id}_{\operatorname{Mc}(u)} - \psi \circ \varphi$ is null homotopic. To this end, we define

$$r^i\colon \operatorname{Mc}(u)^i = X^{i+2} \oplus Y^{i+1} \oplus X^{i+1} \oplus Z^i \xrightarrow{\begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}} X^{i+1} \oplus Y^i \oplus X^i \oplus Z^{i-1} = \operatorname{Mc}(u)^{i-1},$$
$$(x, y, x', z) \longmapsto (x', 0, 0, 0).$$

Now, we compute

$$\begin{aligned}
&r^{i+1} d^i_{\operatorname{Mc}(u)} + d_{\operatorname{Mc}(u)} r^i \\
&\quad= \begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} d & 0 & 0 & 0 \\ -f & -d & 0 & 0 \\ \mathrm{id} & 0 & -d & 0 \\ 0 & g & gf & d \end{pmatrix} + \begin{pmatrix} d & 0 & 0 & 0 \\ -f & -d & 0 & 0 \\ \mathrm{id} & 0 & -d & 0 \\ 0 & g & gf & d \end{pmatrix} \begin{pmatrix} 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad= \begin{pmatrix} \mathrm{id} & 0 & -d & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & d & 0 \\ 0 & 0 & -f & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 & 0 & 0 \\ 0 & 0 & -f & 0 \\ 0 & 0 & \mathrm{id} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \\
&\quad= \mathrm{id}_{\operatorname{Mc}(u)} - \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \mathrm{id} & f & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} = \mathrm{id}_{\operatorname{Mc}(u)} - \begin{pmatrix} 0 & 0 \\ \mathrm{id} & 0 \\ 0 & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 & \mathrm{id} & f & 0 \\ 0 & 0 & 0 & \mathrm{id} \end{pmatrix} \\
&\quad= \mathrm{id}_{\operatorname{Mc}(u)} - \psi \circ \varphi,
\end{aligned}$$

which shows that φ is a homotopy equivalence with homotopy inverse ψ.

Finally, we need to show that the diagram (4.4) commutes. From the definition it is obvious that the left and middle squares commute (already in $\mathsf{C}(\mathcal{A})$). Moreover, we have $\iota_f[1] \circ \pi_g = \pi_u \circ \psi$ in $\mathsf{C}(\mathcal{A})$, from which we deduce (using that φ is the homotopy inverse of ψ) that also the right square commutes in $\mathsf{K}(\mathcal{A})$.

Step 3: We need to check that the diagram (4.3) commutes, but this is immediate from the construction, since all squares and triangles commute already in $\mathsf{C}(\mathcal{A})$:

$$\begin{aligned}
\pi_{gf} \circ u &= \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \pi_f, \\
u \circ \iota_f &= \begin{pmatrix} \mathrm{id} & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 \\ g \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} \circ g = \iota_{gf} \circ g, \\
f[1] \circ \pi_{gf} &= f \circ \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} = \begin{pmatrix} f & 0 \end{pmatrix} = \begin{pmatrix} \mathrm{id} & 0 \end{pmatrix} \begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix} = \pi_g \circ v, \\
v \circ \iota_{gf} &= \begin{pmatrix} f & 0 \\ 0 & \mathrm{id} \end{pmatrix} \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathrm{id} \end{pmatrix} = \iota_g.
\end{aligned}$$

This finishes the verification of (T3) in the special case.

We now handle the general case. Observe first that, if $f\colon X \to Y$ is a morphism of complexes and $X \xrightarrow{f} Y \xrightarrow{f'} Y/X \xrightarrow{f''} X[1]$ is any distinguished triangle, then there exists an isomorphism $\varphi\colon Y/X \xrightarrow{\sim} \mathrm{Mc}(f)$ in $\mathsf{K}(\mathcal{A})$ making the diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, equal] & Y \arrow[r, "f'"] \arrow[d, equal] & Y/X \arrow[r, "f''"] \arrow[d, "\varphi", "\sim"'] & X[1] \arrow[d, equal] \\
X \arrow[r, "f"'] & Y \arrow[r, "\iota_f"'] & \mathrm{Mc}(f) \arrow[r, "\pi_f"'] & X[1]
\end{tikzcd}$$

commutative. Indeed, by definition the distinguished triangle (f, f', f'') is isomorphic to a triangle of the form $X' \xrightarrow{f_0} Y' \to \mathrm{Mc}(f_0) \to X'[1]$, and then Proposition 2.13(iii) provides the map $\varphi\colon Y/X \to \mathrm{Mc}(f)$ making the whole diagram commute. Now Proposition 4.7(iii) shows that φ is an isomorphism.

Let now

$$\begin{gathered}
X \xrightarrow{f} Y \xrightarrow{f'} Y/X \xrightarrow{f''} X[1] \\
Y \xrightarrow{g} Z \xrightarrow{g'} Z/Y \xrightarrow{g''} Y[1] \\
X \xrightarrow{h} Z \xrightarrow{h'} Z/X \xrightarrow{h''} X[1]
\end{gathered}$$

be distinguished triangles in $\mathsf{K}(\mathcal{A})$ such that $h = g \circ f$ in $\mathsf{K}(\mathcal{A})$. Up to changing h by a homotopic map we may assume that $h = g \circ f$ already in $\mathsf{C}(\mathcal{A})$. By the observation above we get the following isomorphisms of distinguished triangles:

$$\begin{aligned}
(\mathrm{id}_X, \mathrm{id}_Y, \varphi)&\colon (f, f', f'') \xrightarrow{\sim} (f, \iota_f, \pi_f), \\
(\mathrm{id}_Y, \mathrm{id}_Z, \gamma)&\colon (g, g', g'') \xrightarrow{\sim} (g, \iota_g, \pi_g), \\
(\mathrm{id}_X, \mathrm{id}_Z, \eta)&\colon (h, h', h'') \xrightarrow{\sim} (h, \iota_h, \pi_h).
\end{aligned}$$

By what we have proved above, we find maps $u\colon \mathrm{Mc}(f) \to \mathrm{Mc}(h)$ and $v\colon \mathrm{Mc}(h) \to \mathrm{Mc}(g)$ such that

$$\mathrm{Mc}(f) \xrightarrow{u} \mathrm{Mc}(h) \xrightarrow{v} \mathrm{Mc}(g) \xrightarrow{\iota_f[1] \circ \pi_g} \mathrm{Mc}(f)[1]$$

is a distinguished triangle. We obtain an isomorphism of triangles

$$\begin{tikzcd}
Y/X \arrow[r, "\widetilde{u}"] \arrow[d, "\varphi"', "\sim"] & Z/X \arrow[r, "\widetilde{v}"] \arrow[d, "\sim"', "\eta"] & Z/Y \arrow[r, "f'[1]\circ g''"] \arrow[d, "\sim"', "\gamma"] & Y/X[1] \arrow[d, "\sim"', "\varphi[1]"] \\
\mathrm{Mc}(f) \arrow[r, "u"'] & \mathrm{Mc}(h) \arrow[r, "v"'] & \mathrm{Mc}(g) \arrow[r, "\iota_f[1]\circ\pi_g"'] & \mathrm{Mc}(f)[1],
\end{tikzcd}$$

where $\widetilde{u} \coloneqq \eta^{-1}u\varphi$ and $\widetilde{v} \coloneqq \gamma^{-1}v\eta$. It remains to check that the diagram

$$\begin{tikzcd}
 & & Y/X \arrow[dr, dashed, "\widetilde{u}"] \arrow[rr, bend left, "f''"] & & X[1] \arrow[dr, "f[1]"] & & \\
 & Y \arrow[ur, "f'"] \arrow[dr, "g"] & & Z/X \arrow[ur, "h''"] \arrow[dr, dashed, "\widetilde{v}"] & & Y[1] \arrow[dr, "f'[1]"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Z \arrow[ur, "h'"] \arrow[rr, bend right, "g'"'] & & Z/Y \arrow[ur, "g''"] \arrow[rr, bend right, "f'[1]\circ g''"'] & & \mathrm{Mc}(f)[1]
\end{tikzcd}$$

commutes. To this end, we compute

$$\begin{aligned}
\widetilde{u}f' &= \eta^{-1}u\varphi f' = \eta^{-1}u\iota_f = \eta^{-1}\iota_h g = h'g, \\
h''\widetilde{u} &= \pi_h\eta\eta^{-1}u\varphi = \pi_h u\varphi = \pi_f\varphi = f'', \\
g''\widetilde{v} &= \pi_g\gamma\gamma^{-1}v\eta = \pi_g v\eta = f[1]\circ\pi_h\eta = f[1]\circ h'', \\
\widetilde{v}h' &= \gamma^{-1}v\eta h' = \gamma^{-1}v\iota_h = \gamma^{-1}\iota_g = g'.
\end{aligned}$$

We conclude that $\mathsf{K}(\mathcal{A})$ is a triangulated category. □

Lemma 4.13 (3 × 3 lemma)**.** *Let $(\mathcal{C}, T)$ be a triangulated category. Consider a diagram with solid arrows*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[r, "g"] \arrow[d, "y"] & Z \arrow[r, "h"] \arrow[d, dashed, "z"] & T(X) \arrow[d, "T(x)"] \\
X' \arrow[r, "f'"] \arrow[d, "x'"'] & Y' \arrow[r, "g'"] \arrow[d, "y'"] & Z' \arrow[r, "h'"] \arrow[d, dashed, "z'"] & T(X') \arrow[d, "T(x')"] \\
X'' \arrow[r, dashed, "f''"] \arrow[d, "x''"'] & Y'' \arrow[r, dashed, "g''"] \arrow[d, "y''"] & Z'' \arrow[r, dashed, "h''"] \arrow[d, dashed, "z''"] \arrow[dr, phantom, "-"] & T(X'') \arrow[d, "T(x'')"] \\
T(X) \arrow[r, "T(f)"'] & T(Y) \arrow[r, "T(g)"'] & T(Z) \arrow[r, "T(h)"'] & T^2(X),
\end{tikzcd}$$

where the upper left square commutes and the first two rows and columns are distinguished triangles. Then there exist the dashed arrows such that the third row and column are distinguished triangles and all squares commute, except for the one marked '−', which anti-commutes.

Proof. By (T1) we may embed $y \circ f = f' \circ x$ in a distinguished triangle $X \xrightarrow{yf} Y' \xrightarrow{p} A \xrightarrow{q} T(X)$. We apply (T3) to the two decompositions $y \circ f = f' \circ x$ and obtain two commutative diagrams

$$\begin{tikzcd}
 & & Z \arrow[rr, bend left, "h"] \arrow[dr, dashed, "s"] & & T(X) \arrow[dr, "T(f)"] & & \\
 & Y \arrow[ur, "g"] \arrow[dr, "y"] & & A \arrow[ur, "q"] \arrow[dr, dashed, "t"] & & T(Y) \arrow[dr, "T(g)"] & \\
X \arrow[ur, "f"] \arrow[rr, bend right, "yf"'] & & Y' \arrow[ur, "p"] \arrow[rr, bend right, "y'"'] & & Y'' \arrow[ur, "y''"] \arrow[rr, bend right, "T(g')y''"'] & & T(Z)
\end{tikzcd}$$

and

$$\begin{tikzcd}
 & & X'' \arrow[rr, bend left, "x''"] \arrow[dr, dashed, "u"] & & T(X) \arrow[dr, "T(x)"] & & \\
 & X' \arrow[ur, "x'"] \arrow[dr, "f'"] & & A \arrow[ur, "q"] \arrow[dr, dashed, "v"] & & T(X') \arrow[dr, "T(x')"] & \\
X \arrow[ur, "x"] \arrow[rr, bend right, "f'x"'] & & Y' \arrow[ur, "p"] \arrow[rr, bend right, "g'"'] & & Z' \arrow[ur, "h'"] \arrow[rr, bend right, "T(x')h'"'] & & T(X''),
\end{tikzcd}$$

where

$$Z \xrightarrow{s} A \xrightarrow{t} Y'' \xrightarrow{T(g')y''} T(Z),$$
$$X'' \xrightarrow{u} A \xrightarrow{v} Z' \xrightarrow{T(x')h'} T(X'')$$

are distinguished triangles. By (T1) we may embed $z \coloneqq vs\colon Z \to Z'$ in a distinguished triangle $Z \xrightarrow{z} Z' \xrightarrow{z'} Z'' \xrightarrow{z''} T(X)$. Applying (T3) to the decomposition $z = vs$, we obtain a commutative diagram

$$\begin{tikzcd}
 & & Y'' \arrow[rr, bend left, "T(g)y''"] \arrow[dr, dashed, "g''"] & & T(Z) \arrow[dr, "T(s)"] & & \\
 & A \arrow[ur, "t"] \arrow[dr, "v"] & & Z'' \arrow[ur, "z''"] \arrow[dr, dashed, "h''"] & & T(A) \arrow[dr, "T(t)"] & \\
Z \arrow[ur, "s"] \arrow[rr, bend right, "z"'] & & Z' \arrow[ur, "z'"] \arrow[rr, bend right, "T(x')h'"'] & & T(X'') \arrow[ur, "-T(u)"] \arrow[rr, bend right, "-T(tu)"'] & & T(Y''),
\end{tikzcd}$$

where $Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} T(X'') \xrightarrow{-T(tu)} T(Y'')$ is a distinguished triangle. Putting $f'' := tu$, we obtain by Proposition 4.10 a distinguished triangle

$$X'' \xrightarrow{f''} Y'' \xrightarrow{g''} Z'' \xrightarrow{h''} T(X'').$$

This finishes the construction of the diagram, in which the first three rows and columns are distinguished triangles. It remains to check the commutativity of the squares:

$$\begin{aligned}
zg &= vsg = vpy = g'y,\\
h'z &= h'vs = T(x)qs = T(x)h,\\
f''x' &= tux' = tpf' = y'f',\\
z'g' &= z'vp = g''tp = g''y',\\
h''z' &= T(x')h',\\
y''f'' &= y''tu = T(f)qu = T(f)x'',\\
z''g'' &= T(g)y'',\\
T(x'')h'' &= T(q)T(u)h'' = -T(qs)z'' = -T(h)z''.
\end{aligned}$$

□

§5. Some properties of triangulated categories

Lemma 5.1. *Let $(\mathcal{C}, T)$ be a triangulated category. Consider a diagram (with solid arrows)*

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "x"'] & Y \arrow[r, "g"] \arrow[d, "y"] & Z \arrow[r, "h"] \arrow[d, dashed, "z"] & T(X) \arrow[d, "T(x)"] \\
X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X')
\end{tikzcd}$$

where the left square commutes and the rows are distinguished triangles. Suppose that one of the following conditions is satisfied: (a) $\mathrm{Hom}_{\mathcal{C}}(T(X), Z') = 0$, or (b) $\mathrm{Hom}_{\mathcal{C}}(Z, Y') = 0$. Then there exists a unique morphism $z \colon Z \to Z'$ making the whole diagram commutative.

Proof. Suppose that $\mathrm{Hom}_{\mathcal{C}}(T(X), Z') = 0$. Let $z_1, z_2 \colon Z \to Z'$ be two morphisms making the diagram commute, so that $g^*(z_1) = z_1 g = g'y = z_2 g = g^*(z_2)$. By Proposition 4.7(ii) the sequence

$$0 = \mathrm{Hom}_{\mathcal{C}}(T(X), Z') \to \mathrm{Hom}_{\mathcal{C}}(Z, Z') \xrightarrow{g^*} \mathrm{Hom}_{\mathcal{C}}(Y, Z').$$

is exact, from which we deduce $z_1 = z_2$. If $\mathrm{Hom}_{\mathcal{C}}(Z, Y') = 0$, then we argue similarly using the exact sequence $0 = \mathrm{Hom}_{\mathcal{C}}(Z, Y') \to \mathrm{Hom}_{\mathcal{C}}(Z, Z') \xrightarrow{h'_*} \mathrm{Hom}_{\mathcal{C}}(Z, T(X'))$ instead. □

Proposition 5.2. *Let $(\mathcal{C}, T)$ be a triangulated category and consider two distinguished triangles*

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X),$$
$$X' \xrightarrow{u'} Y' \xrightarrow{v'} Z' \xrightarrow{w'} T(X').$$

Then the triangle $X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{v \oplus v'} Z \oplus Z' \xrightarrow{w \oplus w'} T(X) \oplus T(X')$ is distinguished.

Proof. By (T1) there exists a distinguished triangle

$$X \oplus X' \xrightarrow{u \oplus u'} Y \oplus Y' \xrightarrow{f} C \xrightarrow{g} T(X \oplus X').$$

Now, Proposition 4.4 provides two morphisms $\zeta\colon Z \to C$ and $\zeta'\colon Z' \to C$ together with morphisms

$$\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "i_X"'] & Y \arrow[r, "v"] \arrow[d, "i_Y"] & Z \arrow[r, "w"] \arrow[d, "\zeta"] & T(X) \arrow[d, "T(\iota_X)"] \\
X \oplus X' \arrow[r, "u \oplus u'"] & Y \oplus Y' \arrow[r, "f"] & C \arrow[r, "g"] & T(X \oplus X') \\
X' \arrow[r, "u'"'] \arrow[u, "i_{X'}"] & Y' \arrow[r, "v'"'] \arrow[u, "\iota_{Y'}"'] & Z' \arrow[r, "w'"'] \arrow[u, "\zeta'"'] & T(X') \arrow[u, "T(\iota_{X'})"']
\end{tikzcd}$$

of distinguished triangles. We obtain a commutative diagram

$$\begin{tikzcd}
X \oplus X' \arrow[r, "u \oplus u'"] \arrow[d, equal] & Y \oplus Y' \arrow[r, "v \oplus v'"] \arrow[d, equal] & Z \oplus Z' \arrow[r, "w \oplus w'"] \arrow[d, "{(\zeta, \zeta')}"] & T(X) \oplus T(X') \arrow[d, "\sim"] \\
X \oplus X' \arrow[r, "u \oplus u'"'] & Y \oplus Y' \arrow[r, "f"'] & C \arrow[r, "g"'] & T(X \oplus X').
\end{tikzcd}$$

of triangles. It suffices to show that $(\zeta, \zeta')\colon Z \oplus Z' \to C$ is an isomorphism. Applying $\mathrm{Hom}_{\mathcal{C}}(M, -)$ for varying $M \in \mathcal{C}$, we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(T(Y \oplus Y'), M) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(T(X \oplus X'), M) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(C, M) \arrow[r] \arrow[d, "{(\zeta, \zeta')^*}"] & \mathrm{Hom}(Y \oplus Y', M) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(X \oplus X', M) \arrow[d, equal] \\
\mathrm{Hom}(T(Y) \oplus T(Y'), M) \arrow[r] & \mathrm{Hom}(T(X) \oplus T(X'), M) \arrow[r] & \mathrm{Hom}(Z \oplus Z', M) \arrow[r] & \mathrm{Hom}(Y \oplus Y', M) \arrow[r] & \mathrm{Hom}(X \oplus X', M)
\end{tikzcd}$$

of abelian groups with exact rows. By the five lemma, the map $(\zeta, \zeta')^*$ is an isomorphism. By the Yoneda lemma, (ζ, ζ') is an isomorphism, which finishes the proof. □

Proposition 5.3. *Let $(\mathcal{C}, T)$ be a triangulated category. Let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ be a distinguished triangle. Then:*

(i) u is an isomorphism if and only if $Z = 0$.

(ii) The triangle $X \xrightarrow{i_X} X \oplus Z \xrightarrow{p_Z} Z \xrightarrow{0} T(X)$ is distinguished.

(iii) If $w = 0$, then the triangle (u, v, w) is isomorphic to $(i_X, p_Z, 0)$. In this case we say that the triangle (u, v, w) splits*.*

Proof. Part (i) is exactly as in Exercise 3.2.

Let us prove (ii). The triangle $(i_X, p_Z, 0)$ is the direct sum of the distinguished triangles $X \xrightarrow{\mathrm{id}} X \to 0 \to T(X)$ and $0 \to Z \xrightarrow{\mathrm{id}} Z \to 0$ (cf. (T1), (T2)). Hence, Proposition 5.2 shows that $(i_X, p_Z, 0)$ is distinguished.

We prove (iii), so assume $w = 0$. Then Proposition 4.4 (applied to appropriately shifted triangles) yields a morphism

$$\begin{tikzcd} X \arrow[r, "i_X"] \arrow[d, equal] & X \oplus Z \arrow[r, "p_Z"] \arrow[d, dashed, "f"] & Z \arrow[r, "0"] \arrow[d, equal] & T(X) \arrow[d, equal] \\ X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "0"'] & T(X) \end{tikzcd}$$

of distinguished trangles, and Proposition 4.7(iii) shows that f is an isomorphism. $\square$

Corollary 5.4. *Let $(\mathcal{C}, T)$ be a triangulated category. Then all monomorphisms and epimorphisms split. In other words, $\mathcal{C}$ is* semi-simple.

Proof. Let $u\colon X \to Y$ be a monomorphism and let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ be a distinguished triangle provided by (T1). By (T2) the triangle $(v, w, -T(u))$ is distinguished. Since T is an equivalence of categories, $T(u)$ is a monomorphism. Since also $-T(u) \circ w = 0$ by Proposition 4.7(i), we conclude that $w = 0$. Now Proposition 5.3(iii) shows that the triangle (u, v, w) splits, which proves the assertion. The argument for epimorphisms is analogous. $\square$

Example 5.5. Let $\mathcal{A}$ be an additive category such that $\mathsf{K}(\mathcal{A})$ is abelian. Then $\mathcal{A}$ and $\mathsf{K}(\mathcal{A})$ are semi-simple.

In particular, $\mathsf{K}(\mathsf{Ab})$ is not abelian.

Proof. Following the strategy in [tu], we proceed in several steps.

Step 1: Let $\mathcal{C}$ be an abelian category. Then $\mathcal{C}$ is semi-simple if and only if every morphism $f\colon A \to B$ admits a *pseudo-inverse*, *i.e.*, a map $g\colon B \to A$ such that $fgf = f$ and $gfg = g$.

Suppose that every morphism admits a pseudo-inverse. Let $u\colon A \hookrightarrow B$ be a monomorphism with pseudo-inverse $v\colon B \to A$. Since u is a monomorphism, the equality $uvu = u$ implies $vu = \mathrm{id}_A$, hence $\mathcal{C}$ is semi-simple. Conversely, suppose that $\mathcal{C}$ is semi-simple. Let $f\colon A \to B$ be a morphism. We factor $f = i \circ p$, where $p\colon A \twoheadrightarrow I$ is an epimorphism and $i\colon I \hookrightarrow B$ is a monomorphism. Since $\mathcal{C}$ is semi-simple, there exist splittings $s\colon I \hookrightarrow A$ and $\pi\colon B \twoheadrightarrow I$ with $ps = \mathrm{id}_I = \pi i$. Putting $g := s \circ \pi\colon B \to A$, we compute $gfg = s\pi ips\pi = s\pi = g$ and $fgf = ips\pi ip = ip = f$, as desired.

Step 2: Let $\mathcal{C}$ be an abelian semi-simple category and $\mathcal{A} \subseteq \mathcal{C}$ a full subcategory. Then $\mathcal{A}$ is semi-simple.

By Step 1 it suffices to check that every morphism in $\mathcal{A}$ admits a pseudo-inverse, which is clear since a pseudo-inverse exists in $\mathcal{C}$ and $\mathcal{A} \subseteq \mathcal{C}$ is full.

Step 3: Proof of the statement. If $\mathsf{K}(\mathcal{A})$ is abelian, then Corollary 5.4 shows that $\mathsf{K}(\mathcal{A})$ is semi-simple. Now $\mathcal{A} \subseteq \mathsf{K}(\mathcal{A})$ is a full subcategory, and hence semi-simple by Step 2. $\square$

§6. Exact functors and triangulated subcategories

Definition 6.1. Let $(\mathcal{C}, T)$, $(\mathcal{D}, S)$ be triangulated categories. An *exact functor* $(\mathcal{C}, T) \to (\mathcal{D}, S)$ consists of a pair (F, ξ), where $F\colon \mathcal{C} \to \mathcal{D}$ is a functor and $\xi\colon FT \xrightarrow{\sim} SF$ is a natural isomorphism, such that for every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ in $\mathcal{C}$ the triangle

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \xrightarrow{\xi_X F(w)} SF(X)$$

in $\mathcal{D}$ is distinguished. We will often denote an exact functor by F instead of (F, ξ).

A natural transformation $\mu\colon (F, \xi) \to (G, \zeta)$ between exact functors is called *exact* if $S\mu \circ \xi = \zeta \circ \mu T$ as natural transformations $FT \to SG$. We denote by

$$\operatorname{Fun}^{\triangle}(\mathcal{C}, \mathcal{D})$$

the category of exact functors and exact natural transformations (see Exercise 6.2).

Exercise 6.2. Let $H\colon (\mathcal{B}, U) \to (\mathcal{C}, T)$, $F, F', F''\colon (\mathcal{C}, T) \to (\mathcal{D}, S)$ and $G\colon (\mathcal{D}, S) \to (\mathcal{E}, R)$ be exact functors between triangulated categories.

(a) Show that the composition $G \circ F$ is an exact functor.

(b) Let $\mu\colon F \to F'$ and $\mu'\colon F' \to F''$ be exact natural transformations. Show that the following natural transformations are exact: (i) $\mu H\colon F \circ H \to F' \circ H$; (ii) $G\mu\colon G \circ F \to G \circ F'$; (iii) $\mu' \circ \mu\colon F \to F''$.

Lemma 6.3. *Let $F\colon (\mathcal{C}, T) \to (\mathcal{D}, S)$ be an exact functor of triangulated categories. Then F is additive.*

Proof. We first show that $F(0) = 0$. Let $X \in \mathcal{C}$ be arbitrary (*e.g.*, $X = 0$) and consider the distinguished triangle $X \xrightarrow{\mathrm{id}} X \to 0 \to T(X)$ in $\mathcal{C}$. Since F is exact, we obtain a distinguished triangle $F(X) \xrightarrow{F(\mathrm{id}_X)} F(X) \longrightarrow F(0) \longrightarrow SF(X)$. As $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ is an isomorphism, Proposition 5.3(i) shows that $F(0) = 0$.

We next show $F(X \oplus Y) \xrightarrow{\sim} F(X) \oplus F(Y)$ for all fixed $X, Y \in \mathcal{C}$ (via the obvious maps). Note that F preserves zero morphisms, because $F(0) = 0$. Proposition 5.3(ii) provides distinguished triangles $X \xrightarrow{i_X} X \oplus Y \xrightarrow{p_Y} Y \xrightarrow{0} T(X)$ and $F(X) \xrightarrow{i_{F(X)}} F(X) \oplus F(Y) \xrightarrow{p_{F(Y)}} F(Y) \xrightarrow{0} SF(X)$. Consider now the following morphism of distinguished triangles:

$$\begin{array}{ccccccc}
F(X) & \xrightarrow{F(i_X)} & F(X \oplus Y) & \xrightarrow{F(p_Y)} & F(Y) & \xrightarrow{0} & SF(X) \\
\| & & \downarrow & & \| & & \| \\
F(X) & \xrightarrow[i_{F(X)}]{} & F(X) \oplus F(Y) & \xrightarrow[p_{F(Y)}]{} & F(Y) & \xrightarrow[0]{} & SF(X),
\end{array}$$

where the top triangle is distinguished because F is exact. By Proposition 4.7(iii) we conclude that $F(X \oplus Y) \xrightarrow{\sim} F(X) \to F(Y)$ is an isomorphism as desired. $\square$

Example 6.4. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between additive categories. Then the induced functor $\mathsf{K}(F)\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathcal{B})$ (given by applying F termwise to a complex) is exact.

Proof. Since F is additive, we have an induced functor $\mathsf{C}(F)\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathcal{B})$ given by applying F termwise to a complex.[2] If s is a homotopy between morphisms f, g, then clearly $F(s)$ is a homotopy between $F(f)$ and $F(g)$, so that $\mathsf{C}(F)$ descends to a functor $\mathsf{K}(F)$ on the homotopy categories. It is trivial to check $\mathsf{K}(F)(X[1]) = \mathsf{K}(F)(X)[1]$ and $\mathsf{K}(F)(\mathrm{Mc}(f)) = \mathrm{Mc}(\mathsf{K}(F)(f))$ for all complexes X and all morphisms f. Hence $\mathsf{K}(F)$ preserves triangles and is thus exact. $\square$

TODO: The following result is optional.

[2] We need additivity to deduce $F(d^i) \circ F(d^{i-1}) = F(d^i \circ d^{i-1}) = F(0) = 0$.

Proposition 6.5. *Let $(F,\xi)\colon (\mathcal{C},T)\to(\mathcal{D},S)$ be an exact functor of triangulated categories. If F admits a (right or left) adjoint $G\colon \mathcal{D}\to\mathcal{C}$, then G can be promoted to an exact functor such that the unit and counit are exact.*

Proof. Suppose that F admits a right adjoint $G\colon\mathcal{D}\to\mathcal{C}$. The case of a left adjoint is dual. We proceed in several steps. The construction of a natural isomorphism $\zeta\colon GS\xrightarrow{\sim}TG$ such that (G,ζ) is exact will be given in Step 2. We denote the unit by $\eta\colon \mathrm{id}_{\mathcal{C}}\to GF$ and the counit by $\varepsilon\colon FG\to\mathrm{id}_{\mathcal{D}}$.

Step 1: Suppose that $\alpha\colon F_1\xrightarrow{\sim}F_2$ is a natural isomorphism of functors $\mathcal{C}\to\mathcal{D}$ and that F_i admits a right adjoint G_i with unit $\eta_i\colon\mathrm{id}_{\mathcal{C}}\to G_iF_i$ and counit $\varepsilon_i\colon F_iG_i\to\mathrm{id}_{\mathcal{D}}$ $(i=1,2)$. Then passing to the right adjoints yields an isomorphism $\alpha^r\colon G_2\xrightarrow{\sim}G_1$ given explicitly by the composite

$$G_2\xrightarrow{\eta_1G_2}G_1F_1G_2\xrightarrow{G_1\alpha G_2}G_1F_2G_2\xrightarrow{G_1\varepsilon_2}G_1.$$

Indeed, for all $X\in\mathcal{C}$ and $Y\in\mathcal{D}$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{D}}(F_2(X),Y) \arrow[r, "\sim"] \arrow[d, "\alpha_X^*"', "\sim"] & \mathrm{Hom}_{\mathcal{C}}(X,G_2(Y)) \arrow[d, dashed, "\alpha^r_{Y*}"] & \varepsilon_{2Y}\circ F_2(f) \arrow[d, mapsto] & f \arrow[l, mapsto] \arrow[d, dashed, mapsto] \\
\mathrm{Hom}_{\mathcal{D}}(F_1(X),Y) \arrow[r, "\sim"] & \mathrm{Hom}_{\mathcal{C}}(X,G_1(Y)) & \varepsilon_{2Y}\circ F_2(f)\circ\alpha_X \arrow[r, mapsto] & G_1(\varepsilon_{2Y}\circ F_2(f)\circ\alpha_X)\circ\eta_{1X},
\end{tikzcd}$$

and the fact that $G_1\varepsilon_{2Y}\circ G_1F_2(f)\circ G_1(\alpha_X)\circ\eta_{1X}=\alpha^r_Y\circ f$ follows from the commutativity of the diagram

$$\begin{tikzcd}
X \arrow[r, "\eta_{1X}"] \arrow[d, "f"'] & G_1F_1(X) \arrow[r, "G_1\alpha_X"] \arrow[d, "G_1F_1(f)"] & G_1F_2(X) \arrow[d, "G_1F_2(f)"] & \\
G_2(Y) \arrow[r, "\eta_{1G_2(Y)}"'] \arrow[rrr, bend right, "\alpha^r_Y"'] & G_1F_1G_2(Y) \arrow[r, "G_1\alpha_{G_2(Y)}"'] & G_1F_2G_2(Y) \arrow[r, "G_1\varepsilon_{2Y}"'] & G_1(Y).
\end{tikzcd}$$

This shows that the dashed vertical arrow indeed makes the diagram commutative and is an isomorphism. From the Yoneda lemma we deduce that α^r_Y is an isomorphism.

Step 2: We apply Step 1 to the natural isomorphism $\alpha=\xi\colon FT\xrightarrow{\sim}SF$. Note that the unit for the adjunction $FT\dashv T^{-1}G$ is given by the composite $\mathrm{id}_{\mathcal{C}}\xrightarrow{\sim}T^{-1}T\xrightarrow{T^{-1}\eta T}T^{-1}GFT$, and the counit is given by the composite $FTT^{-1}G\xrightarrow{\sim}FG\xrightarrow{\varepsilon}\mathrm{id}_{\mathcal{D}}$. Similarly for the unit and counit of $SF\dashv GS^{-1}$. Hence, letting ζ' denote the composite

$$\zeta'\colon TG\xrightarrow{\eta TG}GFTG\xrightarrow{G\xi G}GSFG\xrightarrow{GS\varepsilon}GS,$$

we obtain from Step 1 that the composite $GS^{-1}\xrightarrow{\sim}T^{-1}TGS^{-1}\xrightarrow{T^{-1}\zeta'S^{-1}}T^{-1}GSS^{-1}\xrightarrow{\sim}T^{-1}G$ is an isomorphism. But since T^{-1}, S^{-1} are equivalences of categories, we deduce that ζ' is an isomorphism. We now put $\zeta:=\zeta'^{-1}\colon GS\xrightarrow{\sim}TG$.

Step 3: The natural transformations $\eta\colon \mathrm{id}_{\mathcal{C}} \to GF$ and $\varepsilon\colon FG \to \mathrm{id}_{\mathcal{D}}$ are exact. Indeed, the commutative diagram

$$\begin{tikzcd}
T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] & GFT \arrow[r, "G\xi"] \arrow[d, "GFT\eta"] & GSF \arrow[d, "GSF\eta"'] \arrow[dr, equal] & \\
TGF \arrow[r, "\eta TGF"'] & GFTGF \arrow[r, "G\xi GF"'] & GSFGF \arrow[r, "GS\varepsilon F"'] & GSF \arrow[lll, bend left, "\zeta F"]
\end{tikzcd}$$

shows that the diagram

$$\begin{tikzcd}
T \arrow[rr, equal] \arrow[d, "\eta T"'] & & T \arrow[d, "T\eta"] \\
GFT \arrow[r, "G\xi"'] & GSF \arrow[r, "\zeta F"'] & TGF
\end{tikzcd}$$

commutes. Hence η is exact. Similarly, the commutative diagram

$$\begin{tikzcd}
FTG \arrow[r, "F\eta TG"] \arrow[dr, equal] & FGFTG \arrow[r, "FG\xi G"] \arrow[d, "\varepsilon FTG"] & FGSFG \arrow[r, "FGS\varepsilon"] \arrow[d, "\varepsilon SFG"] & FGS \arrow[d, "\varepsilon S"] \arrow[lll, bend right, "F\zeta"'] \\
& FTG \arrow[r, "\xi G"'] & SFG \arrow[r, "S\varepsilon"'] & S
\end{tikzcd}$$

shows that the diagram

$$\begin{tikzcd}
FGS \arrow[r, "F\zeta"] \arrow[d, "\varepsilon S"'] & FTG \arrow[r, "\xi G"] & SGF \arrow[d, "S\varepsilon"] \\
S \arrow[rr, equal] & & S
\end{tikzcd}$$

commutes. Hence ε is exact.

Step 4: The functor $(G, \zeta)\colon (\mathcal{D}, S) \to (\mathcal{C}, T)$ is exact. Indeed, let $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} S(X)$ be a distinguished triangle in $\mathcal{D}$. By (T1) there exists a distinguished triangle $G(X) \xrightarrow{G(u)} G(Y) \xrightarrow{a} A \xrightarrow{b} TG(X)$. We need to find an isomorphism $f\colon A \xrightarrow{\sim} G(Z)$ making the diagram

$$\begin{tikzcd}
G(X) \arrow[r, "G(u)"] \arrow[d, equal] & G(Y) \arrow[r, "a"] \arrow[d, equal] & A \arrow[r, "b"] \arrow[d, dashed, "f"] & TG(X) \arrow[d, equal] \\
G(X) \arrow[r, "G(u)"'] & G(Y) \arrow[r, "G(v)"'] & G(Z) \arrow[r, "\zeta_Z \circ G(w)"'] & TG(X)
\end{tikzcd} \tag{6.1}$$

commute. Since F is exact, we obtain from Proposition 4.4 a morphism $g\colon F(A) \to Z$ such that the diagram of distinguished triangles

$$\begin{tikzcd}
FG(X) \arrow[r, "FG(u)"] \arrow[d, "\varepsilon_X"'] & FG(Y) \arrow[r, "F(a)"] \arrow[d, "\varepsilon_X"'] & F(A) \arrow[r, "\xi_{G(X)}F(b)"] \arrow[d, dashed, "g"] & SFG(X) \arrow[d, "S\varepsilon_X"] \\
X \arrow[r, "u"'] & Y \arrow[r, "v"'] & Z \arrow[r, "w"'] & S(X)
\end{tikzcd}$$

commutes. We define $f \coloneqq G(g) \circ \eta_A \colon A \to G(Z)$ to be the morphism which is adjoint to g. We then compute

$$\begin{aligned} f \circ a &= G(g) \circ \eta_A \circ a = G(g) \circ GF(a) \circ \eta_{G(Y)} = G(g \circ F(a)) \circ \eta_{G(Y)} \\ &= G(v \circ \varepsilon_X) \circ \eta_{G(Y)} = G(v) \circ G(\varepsilon_Y) \circ \eta_{G(Y)} = G(v), \end{aligned}$$

using naturality of η and the triangle identity $G\varepsilon \circ \eta G = \mathrm{id}_G$, and

$$\begin{aligned} \zeta_Z \circ G(w) \circ f &= \zeta_Z \circ G(w) \circ G(g) \circ \eta_A = \zeta_Z \circ G(w \circ g) \circ \eta_A \\ &= \zeta_Z \circ G(S\varepsilon_X \circ \xi_{G(X)} \circ F(b)) \circ \eta_A = \zeta_Z \circ GS(\varepsilon_X) \circ G(\xi_{G(X)}) \circ GF(b) \circ \eta_A \\ &= TG(\varepsilon_X) \circ \zeta_{FG(Z)} \circ G(\xi_{G(X)}) \circ \eta_{TG(X)} \circ b = TG(\varepsilon_X) \circ T(\eta_{G(X)}) \circ b = b, \end{aligned}$$

using naturality of ζ and that η is exact (for the penultimate step) by Step 3. Hence the diagram (6.1) commutes.

It remains to show that $f \colon A \xrightarrow{\sim} G(Z)$ is an isomorphism. For any $W \in \mathcal{C}$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}(W, G(X)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(W, G(Y)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(W, A) \arrow[r] \arrow[d, "f_*"] & \mathrm{Hom}(W, TG(X)) \arrow[r] \arrow[d, equal] & \mathrm{Hom}(W, TG(Y)) \arrow[d, equal] \\
\mathrm{Hom}(W, G(X)) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(W, G(Y)) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(W, G(Z)) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(W, TG(X)) \arrow[r] \arrow[d, "\sim"] & \mathrm{Hom}(W, TG(Y)) \arrow[d, "\sim"] \\
\mathrm{Hom}(F(W), X) \arrow[r] & \mathrm{Hom}(F(W), Y) \arrow[r] & \mathrm{Hom}(F(W), Z) \arrow[r] & \mathrm{Hom}(F(W), S(X)) \arrow[r] & \mathrm{Hom}(F(W), S(Y))
\end{tikzcd}$$

where the first and last row are exact by Proposition 4.7(i). We deduce that also the second row is exact, and then the five lemma shows that f_* is an isomorphism. From the Yoneda lemma we conclude that f is an isomorphism as desired. This finishes the proof that (G, ζ) is exact. $\square$

Definition 6.6. Let $(\mathcal{C}, T)$ be a triangulated category. A full subcategory $\mathcal{C}' \subseteq \mathcal{C}$ is called *triangulated* if the following properties are satisfied:

(a) For $X \in \mathcal{C}'$ if and only if $T(X) \in \mathcal{C}'$.

(b) If $X \to Y \to Z \to T(X)$ is a distinguished triangle in $\mathcal{C}$ such that $X, Y \in \mathcal{C}'$, then $Z \in \mathcal{C}'$.

Moreover, the subcategory $\mathcal{C}'$ is called *thick* if it is closed under direct summands, *i.e.*, if $X = X_1 \oplus X_1$ lies in $\mathcal{C}'$, then so do X_1 and X_2. A *thick closure* of $\mathcal{C}'$ is the smallest thick triangulated subcategory $\mathcal{C}'^{\oplus} \subseteq \mathcal{C}$ containing $\mathcal{C}'$.

Remark 6.7. Let $\mathcal{C}'$ be a triangulated subcategory of $(\mathcal{C}, T)$. The following observations are immediate:

(i) $\mathcal{C}' \subseteq \mathcal{C}$ is *strictly full*: If $u \colon X \xrightarrow{\sim} Y$ is an isomorphism with $X \in \mathcal{C}'$, then $Y \in \mathcal{C}'$. (Indeed, $0 \in \mathcal{C}'$ by (T1)(b) and $0 \to X \xrightarrow{u} Y \to T(0)$ is distinguished by (T1) and Proposition 4.10.)

(ii) The functor $T \colon \mathcal{C}' \xrightarrow{\sim} \mathcal{C}'$ is an equivalence of categories. (For essential surjectivity, use (i) and the fact that $T(T^{-1}(X)) \cong X$ for any $X \in \mathcal{C}'$.)

(iii) If $X \to Y \to Z \to T(X)$ is a distinguished triangle in $\mathcal{C}$ and two out of X, Y, Z are in $\mathcal{C}'$, then so is the third.

(iv) $(\mathcal{C}', T)$ is a triangulated category (in particular additive) and the inclusion $(\mathcal{C}', T) \to (\mathcal{C}, T)$ is an exact functor.[3]

§7. The opposite triangulated category

Let $(\mathcal{C}, T)$ be a triangulated category. Fix an inverse $T^{-1}\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ of T. We want to put a triangulated structure on the opposite category $(\mathcal{C}^{\mathrm{op}}, T^{-1})$.

Observation 7.1. Since T is an equivalence of categories, its inverse T^{-1} is a left adjoint of T, meaning that we have a natural (in X and Y) isomorphism

$$\begin{aligned} \operatorname{Hom}_{\mathcal{C}}(X, T(Y)) &\xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(T^{-1}(X), Y), \\ w &\longmapsto \tilde{w}, \\ \tilde{v} &\longleftarrow\!\shortmid v. \end{aligned} \tag{7.1}$$

Concretely, fix the natural isomorphism $\alpha\colon T^{-1}T \xrightarrow{\sim} \mathrm{id}_{\mathcal{C}}$ which corresponds to id_T under the above isomorphism. Then (7.1) is given by $\tilde{w} = \alpha_Y \circ T^{-1}(w)$ for $w\colon X \to T(Y)$. In order to describe the inverse, let $\beta\colon \mathrm{id}_{\mathcal{C}} \to TT^{-1}$ be the morphism corresponding to $\mathrm{id}_{T^{-1}}$ under (7.1). We then have commutative diagrams

$$\begin{tikzcd} T^{-1}(X) \arrow[r, "T^{-1}(\beta_X)"] \arrow[dr, "\mathrm{id}"'] & T^{-1}TT^{-1}(X) \arrow[d, "\alpha_{T^{-1}(X)}"] \\ & T^{-1}(X) \end{tikzcd} \qquad \begin{tikzcd} T(Y) \arrow[r, "\beta_{T(Y)}"] \arrow[dr, "\mathrm{id}"'] & TT^{-1}T(Y) \arrow[d, "T(\alpha_Y)"] \\ & T(Y); \end{tikzcd}$$

indeed, the left triangle commutes by definition of β, and the right triangle commutes, because both $\mathrm{id}_{T(Y)}$ and $T(\alpha_Y) \circ \beta_{T(Y)}$ correspond to α_Y under the isomorphism (7.1).

Then the inverse of (7.1) is given by $\tilde{v} = T(v) \circ \beta$. In particular, as T is an equivalence of categories, the Yoneda lemma implies that β is an isomorphism as well.

To summarize, $T\colon \mathcal{C} \xrightarrow{\sim} \mathcal{C}$ is an adjoint equivalence.

Remark 7.2. Let $(\mathcal{C}, T)$ be a triangulated category. Then a triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ is distinguished if and only if the "unshifted" triangle

$$T^{-1}(Z) \xrightarrow{-\tilde{w}} X \xrightarrow{u} Y \xrightarrow{\beta_Z v} TT^{-1}(Z)$$

is distinguished.

Proof. Note that we have an isomorphism of triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, equal] & Y \arrow[r, "v"] \arrow[d, equal] & Z \arrow[r, "w"] \arrow[d, "\beta_Z"] & T(X) \arrow[d, equal] \\ X \arrow[r, "u"'] & Y \arrow[r, "\beta_Z v"'] & TT^{-1}(Z) \arrow[r, "T(\tilde{w})"'] & T(X), \end{tikzcd}$$

[3] In fact, we could have defined a triangulated subcategory as an exact functor $(\mathcal{C}', S) \to (\mathcal{C}, T)$ whose underlying functor $\mathcal{C}' \hookrightarrow \mathcal{C}$ is fully faithful.

where the bottom triangle is the shift of $(-\tilde{w}, u, \beta_Z v)$. Hence the claim follows from Proposition 4.10. $\square$

Notation 7.3. If $f\colon X \to Y$ is a morphism in a category $\mathcal{C}$, we denote by $f^{\mathrm{op}}\colon Y \to X$ the corresponding morphism in the opposite category $\mathcal{C}^{\mathrm{op}}$. Note that $(f \circ g)^{\mathrm{op}} = g^{\mathrm{op}} \circ f^{\mathrm{op}}$ whenever the composition makes sense.

Definition 7.4. Let $(\mathcal{C}, T)$ be a triangulated category. A triangle

$$X \xrightarrow{u^{\mathrm{op}}} Y \xrightarrow{v^{\mathrm{op}}} Z \xrightarrow{w^{\mathrm{op}}} T^{-1}(X)$$

in $(\mathcal{C}^{\mathrm{op}}, T^{-1})$ is called *distinguished* if the corresponding triangle

$$Z \xrightarrow{v} Y \xrightarrow{u} X \xrightarrow{\tilde{w}} T(Z) \qquad \text{(equivalently, } Y \xrightarrow{u} X \xrightarrow{\tilde{w}} T(Z) \xrightarrow{-T(v)} T(Y))$$

is distinguished in $(\mathcal{C}, T)$.

Proposition 7.5. *Let $(\mathcal{C}, T)$ be a triangulated category. Then the opposite category $(\mathcal{C}^{\mathrm{op}}, T^{-1})$ is triangulated.*

Proof. We verify axiom (T1). Clearly, every triangle isomorphic to a distinguished triangle is distinguished. We need to check that the triangle $X \xrightarrow{\mathrm{id}_X^{\mathrm{op}}} X \to 0 \to T^{-1}(X)$ is distinguished. The corresponding triangle in $\mathcal{C}$ is given by $0 \to X \xrightarrow{\mathrm{id}_X} X \to T(0)$, which is clearly distinguished. Let now $u^{\mathrm{op}}\colon X \to Y$ be a morphism in $\mathcal{C}^{\mathrm{op}}$. Then $u\colon Y \to X$ sits in a distinguished triangle $Y \xrightarrow{u} X \xrightarrow{\tilde{w}} T(Z) \xrightarrow{-T(v)} T(Y)$ of $\mathcal{C}$, where we have used that T is an equivalence and $w \mapsto \tilde{w}$ is bijective. But this means that the corresponding triangle $(u^{\mathrm{op}}, v^{\mathrm{op}}, w^{\mathrm{op}})$ is distinguished in $\mathcal{C}^{\mathrm{op}}$, which finishes the verification of (T1).

We next check axiom (T2), so let $X \xrightarrow{u^{\mathrm{op}}} Y \xrightarrow{v^{\mathrm{op}}} Z \xrightarrow{w^{\mathrm{op}}} T^{-1}(X)$ be a distinguished triangle in $\mathcal{C}^{\mathrm{op}}$. We need to check that the shifted triangle $(v^{\mathrm{op}}, w^{\mathrm{op}}, -T^{-1}(u^{\mathrm{op}}))$ is distinguished. By definition, the triangle $(v, u, \tilde{w})$ is distinguished in $\mathcal{C}$. By Remark 7.2 the triangle $(-w, v, \beta_X u)$ is distinguished, which is isomorphic to $(w, v, -\beta_X u)$ via $(-\mathrm{id}_{T^{-1}(X)}, \mathrm{id}_Z, \mathrm{id}_Y)$. Again by definition, we conclude that the triangle $(w^{\mathrm{op}}, v^{\mathrm{op}}, (-\widetilde{\beta_X u})^{\mathrm{op}})$ is distinguished in $\mathcal{C}^{\mathrm{op}}$. But now the claim follows from the computation $\widetilde{\beta_X u} = \alpha_{T^{-1}(X)} T^{-1}(\beta_X u) = \alpha_{T^{-1}(X)} T^{-1}(\beta_X) T^{-1}(u) = T^{-1}(u)$.

Finally, we need to check the octahedral axiom (T3). So let

$$\begin{aligned} X &\xrightarrow{f^{\mathrm{op}}} Y \xrightarrow{f'^{\mathrm{op}}} Z' \xrightarrow{f''^{\mathrm{op}}} T^{-1}(X), \\ Y &\xrightarrow{g^{\mathrm{op}}} Z \xrightarrow{g'^{\mathrm{op}}} X' \xrightarrow{g''^{\mathrm{op}}} T^{-1}(Y), \\ X &\xrightarrow{h^{\mathrm{op}}} Z \xrightarrow{h'^{\mathrm{op}}} Y' \xrightarrow{h''^{\mathrm{op}}} T^{-1}(X) \end{aligned}$$

be distinguished triangles in $\mathcal{C}^{\mathrm{op}}$, where $h = f \circ g$. We need to find morphisms u^{op}, v^{op} such that

$$Z' \xrightarrow{u^{\mathrm{op}}} Y' \xrightarrow{v^{\mathrm{op}}} X' \xrightarrow{(g'' T^{-1}(f'))^{\mathrm{op}}} T^{-1}(Z') \tag{7.2}$$

is a distinguished triangle and such that the diagram

$$
\begin{tikzcd}
 & & Z' \arrow[rr, bend left, "f''^{\mathrm{op}}"] \arrow[dr, dashed, "u^{\mathrm{op}}"] & & T^{-1}(X) \arrow[dr, "T^{-1}(f)^{\mathrm{op}}"] & & \\
 & Y \arrow[ur, "f'^{\mathrm{op}}"] \arrow[dr, "g^{\mathrm{op}}"] & & Y' \arrow[ur, "h''^{\mathrm{op}}"] \arrow[dr, dashed, "v^{\mathrm{op}}"] & & T^{-1}(Y) \arrow[dr, "T^{-1}(f')^{\mathrm{op}}"] & \\
X \arrow[ur, "f^{\mathrm{op}}"] \arrow[rr, bend right, "h^{\mathrm{op}}"'] & & Z \arrow[ur, "h'^{\mathrm{op}}"] \arrow[rr, bend right, "g'^{\mathrm{op}}"'] & & X' \arrow[ur, "g''^{\mathrm{op}}"] \arrow[rr, bend right, "(g''T^{-1}(f'))^{\mathrm{op}}"'] & & T^{-1}(Z').
\end{tikzcd}
$$

commutes. Concretely, we need to verify the following equalities of morphisms in $\mathcal{C}$:

$$f' \circ u = g \circ h' \quad : Y' \to Y \tag{7.3}$$
$$f'' = u \circ h'' \quad : T^{-1}(X) \to Z' \tag{7.4}$$
$$g' = h' \circ v \quad : X' \to Z \tag{7.5}$$
$$h'' \circ T^{-1}(f) = v \circ g'' \quad : T^{-1}(Y) \to Y'. \tag{7.6}$$

By assumption, we have distinguished triangles

$$
\begin{aligned}
Y \xrightarrow{f} X \xrightarrow{\widetilde{f''}} T(Z') \xrightarrow{-T(f')} T(Y),\\
Z \xrightarrow{g} Y \xrightarrow{\widetilde{g''}} T(X') \xrightarrow{-T(g')} T(Z),\\
Z \xrightarrow{h} X \xrightarrow{\widetilde{h''}} T(Y') \xrightarrow{-T(h')} T(X)
\end{aligned}
$$

in $\mathcal{C}$. By (T3), and since T is an equivalence of categories, there exist morphisms $u\colon Z' \to Y'$ and $v\colon Y' \to X'$ in $\mathcal{C}$ such that

$$T(X') \xrightarrow{T(v)} T(Y') \xrightarrow{T(u)} T(Z') \xrightarrow{-T(\widetilde{g''}f')} T^2(X')$$

is a distinguished triangle in $\mathcal{C}$. Then also $(-T(v), -T(u), -T(\widetilde{g''}f'))$ is distinguished, which arises as the triple shift of the triangle

$$X' \xrightarrow{v} Y' \xrightarrow{u} Z' \xrightarrow{\widetilde{g''}f'} T(X'),$$

which by Proposition 4.10 is itself distinguished. But this means that the triangle (7.2) is distinguished, since $\widetilde{g''} \circ f' = (g'' \circ T^{-1}(f'))^{\sim}$ by naturality of the isomorphism (7.1).

Moreover, we know from (T3) that the diagram

$$
\begin{tikzcd}
 & & T(X') \arrow[rr, bend left, "-T(g')"] \arrow[dr, dashed, "T(v)"] & & T(Z) \arrow[dr, "T(g)"] & & \\
 & Y \arrow[ur, "\widetilde{g''}"] \arrow[dr, "f"] & & T(Y') \arrow[ur, "-T(h')"] \arrow[dr, dashed, "T(u)"] & & T(Y) \arrow[dr, "T(\widetilde{g''})"] & \\
Z \arrow[ur, "g"] \arrow[rr, bend right, "h"'] & & X \arrow[ur, "\widetilde{h''}"] \arrow[rr, bend right, "\widetilde{f''}"'] & & T(Z') \arrow[ur, "-T(f')"] \arrow[rr, bend right, "-T(\widetilde{g''}f')"'] & & T^2(X')
\end{tikzcd}
$$

commutes. We deduce that the following identities of morphisms in $\mathcal{C}$:

$$
\begin{aligned}
-T(f') \circ T(u) &= T(g) \circ \big(-T(h')\big), \\
\widetilde{f''} &= T(u) \circ \widetilde{h''}, \\
-T(g') &= -T(h') \circ T(v), \\
\widetilde{h''} \circ f &= T(v) \circ \widetilde{g''}.
\end{aligned}
$$

These prove the identities (7.3), (7.4), (7.5) and (7.6): Indeed, for the first and third identity this is clear from the fact that T is an equivalence of categories. For the second and fourth identity we use that the isomorphism in (7.1) is natural, so that $T(u) \circ \widetilde{h''} = \widetilde{u \circ h''}$, $\widetilde{h''} \circ f = \big(h'' \circ T^{-1}(f)\big)^{\sim}$ and $T(v) \circ \widetilde{g''} = \widetilde{v \circ g''}$. Hence, $\mathcal{C}^{\mathrm{op}}$ satisfies the axiom (T3).

This finishes the proof that $(\mathcal{C}^{\mathrm{op}}, T^{-1})$ is triangulated. $\square$

Exercise 7.6. Let $(\mathcal{C}, T)$ and $(\mathcal{D}, S)$ be triangulated categories. Check that a contravariant exact functor $(\mathcal{C}, T) \to (\mathcal{D}, S)$ (*i.e.*, an exact functor $(\mathcal{C}^{\mathrm{op}}, T^{-1}) \to (\mathcal{D}, S)$) is given by a pair (F, ξ) consisting of a contravariant functor $F\colon \mathcal{C} \to \mathcal{D}$ and a natural equivalence $\xi\colon F \xrightarrow{\sim} SFT$ such that for every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ in $\mathcal{C}$ the triangle

$$
F(Z) \xrightarrow{F(v)} F(Y) \xrightarrow{F(u)} F(X) \xrightarrow{\xi_X SF(w)} SF(Z)
$$

is distinguished in $\mathcal{D}$.

A. Homotopy pullbacks and homotopy pushouts

Definition A.1. Let $(\mathcal{C}, T)$ be a triangulated category. A commutative square

$$
\begin{tikzcd}
X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
X' \arrow[r, "u'"'] & Y'
\end{tikzcd}
$$

is called a *homotopy cartesian* if there exists a morphism $d\colon Y' \to T(X)$ such that

$$X \xrightarrow{\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right)} Y \oplus X' \xrightarrow{(g,u')} Y' \xrightarrow{d} T(X) \tag{1.1}$$

is a distinguished triangle. To signify that a commutative square is homotopy cartesian, we put a '$\square$' in the center of the diagram. If the above square is homotopy cartesian, we call:

- X a *homotopy pullback* of $X' \xrightarrow{u'} Y' \xleftarrow{g} Y$, and
- Y' a *homotopy pushout* of $X' \xleftarrow{f} X \xrightarrow{u} Y$.

Proposition A.2. *Let $(\mathcal{C}, T)$ be a triangulated category. Let*

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \\ X' \arrow[r, "u'"'] & Y \end{tikzcd}$$

be a homotopy cartesian square.

(i) $\mathcal{C}$ admits all homotopy pullbacks and homotopy pushouts.

(ii) For every commutative diagram with solid arrows

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y \arrow[d, "g"] \arrow[ddr, bend left, "a"] & \\ X' \arrow[r, "u'"] \arrow[drr, bend right, "b"'] & Y' \arrow[dr, dashed, "\exists\psi"] & \\ & & W \end{tikzcd}$$

there exists a map $\psi\colon Y' \to W$ such that $b = \psi u'$ and $a = \psi g$; if moreover $\mathrm{Hom}_{\mathcal{C}}(T(X), W) = 0$, then ψ is unique.

Dually, for all maps $X' \xleftarrow{b} W \xrightarrow{a} Y$ such that $u'b = ga$, there exists a map $\psi\colon W \to X$ such that $u\psi = a$ and $f\psi = b$; if moreover $\mathrm{Hom}_{\mathcal{C}}(W, T^{-1}(Y')) = 0$, then ψ is unique.

(iii) Every distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ can be completed to a morphism of distinguished triangles

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "f"'] \arrow[dr, phantom, "\square"] & Y \arrow[r, "v"] \arrow[d, "g"] & Z \arrow[r, "w"] \arrow[d, equal] & T(X) \arrow[d, "T(f)"] \\ X' \arrow[r, "u'"'] & Y' \arrow[r, dashed, "v'"'] & Z \arrow[r, dashed, "w'"'] & T(X'). \end{tikzcd}$$

(iv) For every commutative diagram with solid arrows

$$\begin{tikzcd} Z \arrow[r, "w"] \arrow[d, "h"', "\sim"] & X \arrow[r, "u"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & Y \arrow[r, "v"] \arrow[d, dashed, "g'"] & T(Z) \arrow[d, "T(h)"] \\ Z' \arrow[r, "w''"'] & X' \arrow[r, "u''"'] & Y'' \arrow[r, "v''"'] & T(Z'), \end{tikzcd}$$

where h is an isomorphism and the rows are distinguished triangles, there exists a map g' making the whole diagram commutative and the middle square homotopy cartesian.

Proof. Part (i) is clear from (T1): Given a diagram $X' \xleftarrow{f} X \xrightarrow{u} Y$, we may complete the morphism $X \xrightarrow{\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right)} Y \oplus X'$ to a triangle as in (1.1).

We next prove (ii). By Proposition 4.7(ii) the functor $\operatorname{Hom}_{\mathcal{C}}(-, W)$ is cohomological. Applying it to the distinguished triangle $X \to Y \oplus X' \to Y' \to T(X)$ yields a long exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(T(X), W) \xrightarrow{d^*} \operatorname{Hom}_{\mathcal{C}}(Y', W) \xrightarrow{\begin{pmatrix} g^* \\ u'^* \end{pmatrix}} \operatorname{Hom}_{\mathcal{C}}(Y \oplus X', W) \xrightarrow{(u^*, -f^*)} \operatorname{Hom}_{\mathcal{C}}(X, W).$$

Now, we have $(u^*, -f^*)\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) = au - bf = 0$. Hence, there exists $\psi \in \operatorname{Hom}_{\mathcal{C}}(Y', W)$ such that $\begin{pmatrix} \psi g \\ \psi u' \end{pmatrix} = \begin{pmatrix} g^* \\ u'^* \end{pmatrix}(\psi) = \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)$. Moreover, if $\operatorname{Hom}_{\mathcal{C}}(T(X), W) = 0$, then ψ is unique. The dual assertion follows similarly using that $\operatorname{Hom}_{\mathcal{C}}(W, -)$ is cohomological.

We now prove (iii). Factor u as the composite $X \xrightarrow{\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right)} Y \oplus X' \xrightarrow{p_Y} Y$, and obtain the following distinguished triangles:

$$X \xrightarrow{\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right)} Y \oplus X' \xrightarrow{(g, u')} Y' \xrightarrow{d} T(X),$$
$$Y \oplus X' \xrightarrow{p_Y} Y \xrightarrow{0} T(X') \xrightarrow{-T(i_{X'})} T(Y \oplus X'),$$
$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X);$$

note that the second triangle is distinguished by Proposition 5.3(ii) and (T2). By (T3) we obtain a distinguished triangle

$$Y' \xrightarrow{v'} Z \xrightarrow{w'} T(X') \xrightarrow{-T(u')} T(Y').$$

Now, applying Proposition 4.10 shows that $X' \xrightarrow{u'} Y' \xrightarrow{v'} Z \xrightarrow{w'} T(X')$ is distinguished. Moreover, we have a commutative diagram

$$\begin{tikzcd}
 & & Y' \arrow[rr, bend left, "d"] \arrow[dr, dashed, "v'"] & & T(X) \arrow[dr, "T(\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right))"] & & \\
 & Y \oplus X' \arrow[ur, "(g,u')"] \arrow[dr, "p_Y"] & & Z \arrow[ur, "w"] \arrow[dr, dashed, "w'"] & & T(Y \oplus X') \arrow[dr, "T((g,u'))"] & \\
X \arrow[ur, "\left(\begin{smallmatrix} u \\ -f \end{smallmatrix}\right)"] \arrow[rr, bend right, "u"'] & & Y \arrow[ur, "v"] \arrow[rr, bend right, "0"'] & & T(X') \arrow[ur, "-T(i_{X'})"] \arrow[rr, bend right, "-T(u')"'] & & T(Y').
\end{tikzcd}$$

In particular, we deduce the relations $v'g = v$ and $w' = T(f)w$. We thus obtain the desired morphism of distinguished triangles.

Finally, let us prove (iv). By (iii) (and a shift of triangles) we obtain the following commutative diagram with solid arrows

$$\begin{tikzcd}
Z \arrow[r, "w"] \arrow[d, equal] & X \arrow[r, "u"] \arrow[d, "f"'] & Y \arrow[r, "v"] \arrow[d, "g"] & T(Z) \arrow[d, equal] \\
Z \arrow[r, "w'"] \arrow[d, "h"', "\sim"] & X' \arrow[r, "u'"] \arrow[d, equal] & Y' \arrow[r, "v'"] \arrow[d, dashed, "\psi"] & T(Z) \arrow[d, "T(h)"] \\
Z' \arrow[r, "w''"'] & X' \arrow[r, "u''"'] & Y'' \arrow[r, "v''"'] & T(Z'),
\end{tikzcd}$$

where the rows are distinguished triangles and the top middle square is homotopy cartesian. By Proposition 4.4 we find a morphism ψ making the whole diagram commutative. Now, ψ is an isomorphism by Proposition 4.7(iii). But then putting $g' \coloneqq \psi \circ g$ proves the assertion. $\square$

Chapter 3

Localization of Categories

The localization of triangulated categories in a special case of the localization of ordinary categories, so we will treat the latter first.

§8. Localization of additive categories

Let us first look at the example of localization of rings.[1]

Example 8.1. Let A be a ring and $S \subseteq A$ a subset. By possibly increasing S, we may assume $1 \in S$ and that $a, b \in S$ implies $ab \in S$. The localization of A at S is a pair $(A[S^{-1}], \iota)$, where $A[S^{-1}]$ is a ring and $\iota\colon A \to A[S^{-1}]$ is a ring map with $\iota(S) \subseteq A[S^{-1}]^\times$, satisfying the following universal property: For every ring R and every ring morphism $f\colon A \to R$ with $f(S) \subseteq R^\times$ there exists a unique ring map $\overline{f}\colon A[S^{-1}] \to R$ making the following diagram commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\iota"'] & R \\ A[S^{-1}]. \arrow[ur, "\overline{f}"'] & \end{tikzcd}$$

In other words: Precomposition with ι induces an injection

$$\mathrm{Hom}_{\mathsf{Ring}}(A[S^{-1}], R) \hookrightarrow \mathrm{Hom}_{\mathsf{Ring}}(A, R)$$

whose image consists of those ring maps $f\colon A \to R$ satisfying $f(S) \subseteq R^\times$.

The notion of localization of categories is analogous, but "one categorical level higher".

Definition 8.2. Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. A *localization of $\mathcal{C}$ with respect to S* consists of a pair $(\mathcal{C}[S^{-1}], Q)$, where $\mathcal{C}[S^{-1}]$ is a category and $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ is a functor such that $Q(s)$ is invertible for every $s \in S$, which satisfies the following universal property: For all categories $\mathcal{D}$ the natural functor

$$\mathrm{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \xrightarrow{Q^*} \mathrm{Fun}(\mathcal{C}, \mathcal{D})$$

[1]Observe that a ring is the same thing as an Ab-enriched category with one object.

given by precomposition with Q induces an equivalence onto the full subcategory $\mathrm{Fun}^S(\mathcal{C},\mathcal{D}) \subseteq \mathrm{Fun}(\mathcal{C},\mathcal{D})$ of those functors $F\colon \mathcal{C} \to \mathcal{D}$ sending each morphism in S to an isomorphism in $\mathcal{D}$.

In other words:

(1) For every functor $F\colon \mathcal{C} \to \mathcal{D}$ sending the morphisms in S to isomorphisms in $\mathcal{D}$, there exists a functor $\overline{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ and a natural isomorphism $\theta\colon F \overset{\sim}{\Longrightarrow} \overline{F} \circ Q$, that is, there is a commutative diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "Q"'] \arrow[dr, phantom, "\overset{\theta}{\cong}"] & \mathcal{D} \\ \mathcal{C}[S^{-1}]; \arrow[ur, "\overline{F}"'] & \end{tikzcd}$$

(2) For all functors $\overline{F}, \overline{G}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ the map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}[S^{-1}],\mathcal{D})}(\overline{F},\overline{G}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\overline{F}\circ Q, \overline{F}\circ Q), \\ \alpha &\longmapsto \alpha Q \end{aligned} \tag{8.1}$$

is bijective.

Remark 8.3. Let $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ be a localization functor and $\mathcal{D}$ a category.

(a) The fully faithfulness of $Q^*\colon \mathrm{Fun}(\mathcal{C}[S^{-1}],\mathcal{D}) \hookrightarrow \mathrm{Fun}(\mathcal{C},\mathcal{D})$ implies that a natural transformation $\alpha\colon \overline{F} \Longrightarrow \overline{G}$ of functors $\mathcal{C}[S^{-1}] \to \mathcal{D}$ is an isomorphism if and only if $\alpha Q\colon \overline{F}Q \Longrightarrow \overline{G}Q$ is an isomorphism of functors $\mathcal{C} \to \mathcal{D}$.

(b) Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, the pair $(\overline{F}, \theta\colon F \overset{\sim}{\Longrightarrow} \overline{F}\circ Q)$ is unique up to unique isomorphism: For every other pair $(\overline{F}', \theta'\colon F \overset{\sim}{\Longrightarrow} \overline{F}' \circ Q)$ we obtain from (2) above a unique natural transformation $\alpha\colon \overline{F} \Longrightarrow \overline{F}'$ such that $\theta' = \alpha Q \circ \theta$. Now, $\alpha Q = \theta' \circ \theta^{-1}$ is an isomorphism, and hence (a) implies that α is an isomorphism as well.

Remark 8.4. (a) The localization $(\mathcal{C}[S^{-1}], Q)$ always exists. A proof is sketched in [GM03, III.2]. However, it may happen that $\mathcal{C}[S^{-1}]$ is not locally small (meaning that the Hom spaces are proper classes). Even when $\mathcal{C}[S^{-1}]$ is locally small, this is usually highly non-trivial to prove.

(b) The localization $(\mathcal{C}[S^{-1}], Q)$ is unique in the following sense: If $(\widetilde{\mathcal{C}[S^{-1}]}, \widetilde{Q})$ is another localization, then there exists an equivalence of categories $\overline{F}\colon \mathcal{C}[S^{-1}] \xrightarrow{\sim} \widetilde{\mathcal{C}[S^{-1}]}$ and a natural isomorphism $\theta\colon \widetilde{Q} \overset{\sim}{\Longrightarrow} \overline{F}Q$. Moreover, the pair $(\overline{F}, \theta)$ is unique up to unique isomorphism.

Proof. By the universal property of $\mathcal{C}[S^{-1}]$, there exists a functor $\overline{F}\colon \mathcal{C}[S^{-1}] \to \widetilde{\mathcal{C}[S^{-1}]}$ and a natural isomorphism $\theta\colon \widetilde{Q} \overset{\sim}{\Longrightarrow} \overline{F}Q$. Similarly, by the universal property of $\widetilde{\mathcal{C}[S^{-1}]}$ there exists a functor $\overline{G}\colon \widetilde{\mathcal{C}[S^{-1}]} \to \mathcal{C}[S^{-1}]$ and a natural isomorphism $\tau\colon Q \overset{\sim}{\Longrightarrow} \overline{G}\widetilde{Q}$. Now we have an isomorphism

$$\overline{G}\theta \circ \tau\colon Q \overset{\sim}{\Longrightarrow} \overline{GF}Q$$

of functors $\mathcal{C} \to \mathcal{C}[S^{-1}]$, and hence by (2) above there exists a unique natural isomorphism $\alpha\colon \mathrm{id}_{\mathcal{C}[S^{-1}]} \overset{\sim}{\Longrightarrow} \overline{GF}$ of functors $\mathcal{C}[S^{-1}] \to \mathcal{C}[S^{-1}]$ such that $\alpha Q = \overline{G}\theta \circ \tau$. Similarly, we have an isomorphism

$$\overline{F}\tau \circ \theta\colon \widetilde{Q} \overset{\sim}{\Longrightarrow} \overline{FG}\widetilde{Q}$$

of functors $\mathcal{C} \to \widetilde{\mathcal{C}[S^{-1}]}$, and hence by (2) above there exists a unique natural isomorphism $\beta\colon \mathrm{id}_{\widetilde{\mathcal{C}[S^{-1}]}} \overset{\sim}{\Longrightarrow} \overline{FG}$ of functors $\widetilde{\mathcal{C}[S^{-1}]} \to \widetilde{\mathcal{C}[S^{-1}]}$ such that $\beta\widetilde{Q} = \overline{F}\tau \circ \theta$. In particular, $\overline{F}$ and $\overline{G}$ are equivalences which are quasi-inverse to each other. The uniqueness of $(\overline{F}, \theta)$ is discussed in Remark 8.3(b). ☐

Example 8.5. Let $\mathcal{A}$ be an abelian category. Consider the category $\mathsf{C}(\mathcal{A})$ of complexes in $\mathcal{A}$ and let qis be the class of quasi-isomorphisms in $\mathsf{C}(\mathcal{A})$. Then

$$\mathsf{D}(\mathcal{A}) \coloneqq \mathsf{C}(\mathcal{A})[\mathrm{qis}^{-1}]$$

is the (unbounded) *derived category* of $\mathcal{A}$. We similarly define the bounded derived categories $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^b(\mathcal{A})$.

Our next immediate goal is to prove that localizations of additive categories are again additive. We follow the outline sketched in [Cu10].

Lemma 8.6. *Let $\mathcal{C}$ be a category and S a class of morphisms in $\mathcal{C}$. Then the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ is essentially surjective.*

Proof. We factor Q as a composite of functors $Q_1\colon \mathcal{C} \twoheadrightarrow \mathcal{C}_S$ and $Q_2\colon \mathcal{C}_S \hookrightarrow \mathcal{C}[S^{-1}]$, where $\mathcal{C}_S$ is the essential image of Q. Note that Q_1 inverts the morphisms in S, and hence the universal property of $\mathcal{C}[S^{-1}]$ provides a functor $R\colon \mathcal{C}[S^{-1}] \to \mathcal{C}_S$ together with a natural isomorphism $\theta\colon Q_1 \overset{\sim}{\Longrightarrow} R\circ Q$. We obtain a natural isomorphism $Q_2\theta\colon Q \overset{\sim}{\Longrightarrow} Q_2\circ R\circ Q$, and by (2) in Definition 8.2 and Remark 8.3(a), we find a unique isomorphism $\varphi\colon \mathrm{id}_{\mathcal{C}[S^{-1}]} \overset{\sim}{\Longrightarrow} Q_2 \circ R$ such that $\varphi Q = Q_2\theta$. We deduce that Q_2, and therefore also $Q = Q_2 \circ Q_1$, is essentially surjective. ☐

Lemma 8.7. *Let $\mathcal{C}$, $\mathcal{D}$ be categories with chosen classes of morphisms S, T, respectively. Suppose that S and T contain the identities. Then there is a natural equivalence of categories*

$$(\mathcal{C} \times \mathcal{D})[(S \times T)^{-1}] \xrightarrow{\sim} \mathcal{C}[S^{-1}] \times \mathcal{D}[T^{-1}]$$

which is compatible with the localization functors.

Proof. Let $\mathcal{E}$ be a category. We have to check that the essential images of the functors

$$\mathrm{Fun}\big((\mathcal{C} \times \mathcal{D})[(S \times T)^{-1}], \mathcal{E}\big) \longhookrightarrow \mathrm{Fun}(\mathcal{C} \times \mathcal{D}, \mathcal{E}) \longhookleftarrow \mathrm{Fun}\big(\mathcal{C}[S^{-1}] \times \mathcal{D}[T^{-1}], \mathcal{E}\big)$$

agree. In other words, we need to show that a functor $F\colon \mathcal{C} \times \mathcal{D} \to \mathcal{E}$ inverts the morphisms in $S \times T$ if and only if it inverts the morphisms in S and T; but this is obvious from our hypothesis that S and T contain the identities (every morphism $(s,t) \in S \times T$ decomposes as $(s, \mathrm{id}) \circ (\mathrm{id}, t)$).

We deduce that the categories $(\mathcal{C} \times \mathcal{D})[(S \times T)^{-1}]$ and $\mathcal{C}[S^{-1}] \times \mathcal{D}[T^{-1}]$ (together with their localization functors) satisfy the same universal property. By Remark 8.4(b) the canonical functor in the assertion is an equivalence of categories. ☐

Lemma 8.8. *Let $\mathcal{C}$, $\mathcal{D}$ be categories with chosen classes of morphisms S, T, respectively. Let $L\colon \mathcal{C} \rightleftarrows \mathcal{D} : R$ be an adjunction such that $L(S) \subseteq T$ and $R(T) \subseteq S$.*

Then there is an induced adjunction $L_S\colon \mathcal{C}[S^{-1}] \rightleftarrows \mathcal{D}[T^{-1}] : R_T$. Moreover:

(i) There are natural isomorphisms $L_S Q_{\mathcal{C}} \cong Q_{\mathcal{D}} L$ and $Q_{\mathcal{C}} R \cong R_T Q_{\mathcal{D}}$ which are obtained from each other by the adjunctions.

(ii) If the unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to RL$ is an isomorphism, then so is the unit $\eta_S\colon \mathrm{id}_{\mathcal{C}[S^{-1}]} \to R_T L_S$. Similarly, if the counit $\varepsilon\colon LR \to \mathrm{id}_{\mathcal{D}}$ is an isomorphism, then so is the counit $\varepsilon_T\colon L_S R_T \to \mathrm{id}_{\mathcal{D}[T^{-1}]}$.

Proof. Recall that an adjunction comes with natural transformations $\varepsilon\colon LR \to \mathrm{id}_{\mathcal{D}}$ and $\eta\colon \mathrm{id}_{\mathcal{C}} \to RL$ such that the following triangles commute:

$$\begin{tikzcd} L \arrow[r, "L\eta"] \arrow[dr, "\mathrm{id}_L"'] & LRL \arrow[d, "\varepsilon L"] \\ & L \end{tikzcd} \qquad \begin{tikzcd} R \arrow[r, "\eta R"] \arrow[dr, "\mathrm{id}_R"'] & RLR \arrow[d, "R\varepsilon"] \\ & L. \end{tikzcd}$$

Now, since $L(S) \subseteq T$, the functor $Q_{\mathcal{D}}L\colon \mathcal{C} \to \mathcal{D}[T^{-1}]$ inverts the morphisms in S. Hence, there exists a unique pair (L_S, φ_L) consisting of a functor $L_S\colon \mathcal{C}[S^{-1}] \to \mathcal{D}[T^{-1}]$ and a natural isomorphism $\varphi_L\colon L_S Q_{\mathcal{C}} \xrightarrow{\sim} Q_{\mathcal{D}}L$. Similarly, there exists a unique pair (R_T, φ_R) consisting of a functor $R_T\colon \mathcal{D}[T^{-1}] \to \mathcal{C}[S^{-1}]$ together with a natural isomorphism $\varphi_R\colon Q_{\mathcal{C}}R \xrightarrow{\sim} R_T Q_{\mathcal{D}}$.

We define $\eta_S\colon \mathrm{id}_{\mathcal{C}[S^{-1}]} \to R_T L_S$ as the image of η under the map

$$\begin{aligned} \mathrm{Hom}(\mathrm{id}_{\mathcal{C}}, RL) \xrightarrow{Q_{\mathcal{C}}} \mathrm{Hom}(Q_{\mathcal{C}}, Q_{\mathcal{C}}RL) &\cong \mathrm{Hom}(Q_{\mathcal{C}}, R_T L_S Q_{\mathcal{C}}) \\ &\cong \mathrm{Hom}(\mathrm{id}_{\mathcal{C}[S^{-1}]}, R_T L_S). \end{aligned}$$

We similarly define $\varepsilon_T\colon L_S R_T \to \mathrm{id}_{\mathcal{D}[T^{-1}]}$. Concretely, this means that the following two diagrams commute:

$$\begin{tikzcd} Q_{\mathcal{C}} \arrow[r, "\eta_S Q_{\mathcal{C}}"] \arrow[d, "Q_{\mathcal{C}}\eta"'] & R_T L_S Q_{\mathcal{C}} \arrow[d, "R_T\varphi_L", "\sim"'] \\ Q_{\mathcal{C}}RL \arrow[r, "\sim", "\varphi_R L"'] & R_T Q_{\mathcal{D}} L \end{tikzcd} \qquad \begin{tikzcd} L_S Q_{\mathcal{C}} R \arrow[r, "L_S\varphi_R", "\sim"'] \arrow[d, "\varphi_L R"', "\sim"] & L_S R_T Q_{\mathcal{D}} \arrow[d, "\varepsilon_T Q_{\mathcal{D}}"] \\ Q_{\mathcal{D}} LR \arrow[r, "Q_{\mathcal{D}}\varepsilon"'] & Q_{\mathcal{D}}. \end{tikzcd} \tag{8.2}$$

From the diagrams we observe that, if η is an isomorphism, then so is $\eta_S Q_{\mathcal{C}}$. As $Q_{\mathcal{C}}$ is a localization functor, it follows that η_S is an isomorphism. Similarly, if ε is an isomorphism, then so is ε_T.

It remains to prove that $(L_S, R_T, \eta_S, \varepsilon_T)$ defines an adjunction, *i.e.*, that $R_T\varepsilon_T \circ \eta_S R_T = \mathrm{id}_{R_T}$ and $\varepsilon_T L_S \circ L_S\eta_S = \mathrm{id}_{L_S}$. By (8.1) these identities can be checked after precomposing with $Q_{\mathcal{D}}$ and $Q_{\mathcal{C}}$, respectively. We only verify the first identity, because the second one is analogous. Consider the diagram

$$\begin{tikzcd} & Q_{\mathcal{C}}RLR \arrow[r, "Q_{\mathcal{C}}R\varepsilon"] \arrow[d, "\varphi_R LR", "\sim"'] & Q_{\mathcal{C}}R \arrow[d, "\varphi_R", "\sim"'] \\ & R_T Q_{\mathcal{D}} LR \arrow[r, "R_T Q_{\mathcal{D}}\varepsilon"] & R_T Q_{\mathcal{D}} \\ Q_{\mathcal{C}}R \arrow[uur, bend left, "Q_{\mathcal{C}}\eta R"] \arrow[r, "\eta_S Q_{\mathcal{C}} R"] \arrow[d, "\varphi_R"', "\sim"] & R_T L_S Q_{\mathcal{C}} R \arrow[u, "R_T\varphi_L R", "\sim"'] \arrow[d, "R_T L_S\varphi_R", "\sim"'] & \\ R_T Q_{\mathcal{D}} \arrow[r, "\eta_S R_T Q_{\mathcal{D}}"'] & R_T L_S R_T Q_{\mathcal{D}}. \arrow[uur, bend right, "R_T\varepsilon_T Q_{\mathcal{D}}"'] & \end{tikzcd}$$

The small squares commute by naturality of η_S and ε, and the "triangles" are the commutative diagrams (8.2). Hence the whole diagram commutes. We thus have

$$(R_T\varepsilon_T Q_{\mathcal{D}}) \circ (\eta_S R_T Q_{\mathcal{D}}) \circ \varphi_R = \varphi_R \circ (Q_{\mathcal{C}}R\varepsilon) \circ (Q_{\mathcal{C}}\eta R) = \varphi_R = (\mathrm{id}_{R_T} Q_{\mathcal{D}}) \circ \varphi_R,$$

where the second identity holds because $R\varepsilon \circ \eta R = \mathrm{id}_R$. As φ_R is an isomorphism, we deduce the right triangle identity for $(L_S, R_T, \eta_S, \varepsilon_T)$. The left triangle identity is analogous and left to the reader.

The assertion that φ_L and φ_R are mates means that we obtain each from the other via

$$\begin{aligned}
\varphi_L &= (\varepsilon_T Q_{\mathcal{D}} L) \circ (L_S \varphi_R L) \circ (L_S Q_{\mathcal{C}} \eta), \\
\varphi_R &= (R_T Q_{\mathcal{D}} \varepsilon) \circ (R_T \varphi_L R) \circ (\eta_S Q_{\mathcal{C}} R).
\end{aligned}$$

(In fact, the two identities are equivalent: Exercise!) Indeed, we have a commutative diagrams

$$\begin{tikzcd}
Q_{\mathcal{C}} R \arrow[r, "\eta_S Q_{\mathcal{C}} R"] \arrow[d, "Q_{\mathcal{C}} \eta R"] \arrow[dd, bend right=60, equal] & R_T L_S Q_{\mathcal{C}} R \arrow[d, "R_T \varphi_L R"] \\
Q_{\mathcal{C}} RLR \arrow[r, "\varphi_R LR"] \arrow[d, "Q_{\mathcal{C}} R\varepsilon"] & R_T Q_{\mathcal{D}} LR \arrow[d, "R_T Q_{\mathcal{D}} \varepsilon"] \\
Q_{\mathcal{C}} R \arrow[r, "\varphi_R"'] & R_T Q_{\mathcal{D}}
\end{tikzcd}
\qquad
\begin{tikzcd}
L_S Q_{\mathcal{C}} \arrow[r, "L Q_{\mathcal{C}} \eta"] \arrow[d, "L_S \eta_S Q_{\mathcal{C}}"] \arrow[dd, bend right=60, equal] & L_S Q_{\mathcal{C}} RL \arrow[d, "L_S \varphi_R L"] \\
L_S R_T L_S Q_{\mathcal{C}} \arrow[r, "L_S R_T \varphi_L"] \arrow[d, "\varepsilon_T L_S Q_{\mathcal{C}}"] & L_S R_T Q_{\mathcal{D}} L \arrow[d, "\varepsilon_T Q_{\mathcal{D}} L"] \\
L_S Q_{\mathcal{C}} \arrow[r, "\varphi_L"'] & Q_{\mathcal{D}} L,
\end{tikzcd}$$

where the top diagrams commute by (8.2) and the bottom diagrams commute because φ_R and ε_T are natural. This proves the claim. ☐

Example 8.9. Let $\mathcal{C}$ be a category and I a set; we will only need the cases $I = \varnothing$ and $I = \{1, 2\}$. Consider the diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^I$ given on objects by $\Delta(C) = (C)_i$ and on morphisms by $\Delta(f) = (f)_i$.

(a) Δ admits a right adjoint $R\colon \mathcal{C}^I \to \mathcal{C}$ if and only if $\mathcal{C}$ admits I-fold products. In this case, R is given by $R(C_i)_i = \prod_{i\in I} C_i$ and the unit $\mathrm{id}_{\mathcal{C}} \to R\Delta$ is given by the diagonal $C \to \prod_{i\in I} C$. Indeed, for all $A \in \mathcal{C}$ we have natural bijections

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(A, R(C_i)_i) &= \mathrm{Hom}_{\mathcal{C}^I}\big((A)_i, (C_i)_i\big) \\
&= \prod_{i\in I} \mathrm{Hom}_{\mathcal{C}}(A, C_i) = \mathrm{Hom}_{\mathcal{C}}\Big(A, \prod_i C_i\Big).
\end{aligned}$$

Hence the claim follows from the Yoneda lemma.

(b) Δ admits a left adjoint $L\colon \mathcal{C}^I \to \mathcal{C}$ if and only if $\mathcal{C}$ admits I-fold coproducts. In this case, L is given by $L(C_i)_i = \coprod_{i\in I} C_i$ and the counit $L\Delta \to \mathrm{id}_{\mathcal{C}}$ is given by the fold map $\coprod_{i\in I} C \to C$.

Proposition 8.10. *Let $\mathcal{C}$ be an additive category. Let S be a class of morphisms which is closed under biproducts. Concretely, this means:*

(a) S contains all identities: for every $C \in \mathcal{C}$ we have $\mathrm{id}_C \in S$;

(b) S is closed under binary biproducts: if $s, t \in S$, then $s \oplus t \in S$.

Then $\mathcal{C}[S^{-1}]$ is an additive category and $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ is additive.

Proof. We first check that $\mathcal{C}[S^{-1}]$ admits biproducts. Let I be a finite set (think of $I = \varnothing$ and $I = \{1, 2\}$). The diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C}^I$ admits a right adjoint R by Example 8.9, which agrees with the biproduct. We put $T = S^I$. Then $\Delta(S) \subseteq T$ and our hypothesis on S ensures that $R(T) \subseteq S$. By Lemma 8.7 we have a commutative diagram

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Delta"] \arrow[d, "Q"'] & \mathcal{C}^I \arrow[d, "Q_{\mathcal{C}^I}"] \arrow[dd, bend left=70, "Q^I"] \\
\mathcal{C}[S^{-1}] \arrow[r, "\Delta"] \arrow[dr, "\Delta"'] & \mathcal{C}^I[T^{-1}] \arrow[d, "\sim"] \\
 & \mathcal{C}[S^{-1}]^I
\end{tikzcd} .$$

By Lemma 8.8 the functor $\Delta\colon \mathcal{C}[S^{-1}] \to \mathcal{C}^I[T^{-1}] \cong \mathcal{C}[S^{-1}]^I$ admits a right adjoint R_T, and the natural transformation $\varphi_R\colon QR \xrightarrow{\sim} R_T Q^I$ (coming from the obvious isomorphism $\Delta Q \xrightarrow{\sim} Q^I \Delta$) is an isomorphism.

Hence, $\mathcal{C}[S^{-1}]$ admits finite products and the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ preserves them. A completely analogous argument shows that $\mathcal{C}[S^{-1}]$ admits finite coproducts and Q preserves them. As a consequence, $\mathcal{C}[S^{-1}]$ has biproducts and the localization functor Q is additive.

It remains to show that for every object $C \in \mathcal{C}[S^{-1}]$ the canonical structure of a commutative monoid on C is a group or, equivalently, the embedding $i\colon \mathsf{CGrp}(\mathcal{C}[S^{-1}]) \hookrightarrow \mathsf{CMon}(\mathcal{C}[S^{-1}]) = \mathcal{C}[S^{-1}]$ is essentially surjective. Observe that Q factors as the composition

$$\mathcal{C} = \mathsf{CGrp}(\mathcal{C}) \longrightarrow \mathsf{CGrp}(\mathcal{C}[S^{-1}]) \overset{i}{\hookrightarrow} \mathcal{C}[S^{-1}].$$

Now, Q is essentially surjective by Lemma 8.6, hence so is i. □

In the next section we will study the following situation.

Exercise 8.11. Let $\mathcal{C}$ be a category and S a class of morphisms of $\mathcal{C}$. Suppose that the localization functor $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ admits a right adjoint $R\colon \mathcal{C}[S^{-1}] \to \mathcal{C}$. Show that R is fully faithful.

§9. Bousfield localizations

One of the most important examples of localizations are Bousfield localizations.

Definition 9.1. Let $Q\colon \mathcal{C} \rightleftarrows \mathcal{D} :R$ be an adjunction.

(a) Q is called a *Bousfield localization* if R is fully faithful (equivalently: the counit $QR \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ is an isomorphism).

(b) A morphism f in $\mathcal{C}$ is called an *R-local equivalence* if $Q(f)$ is an isomorphism in $\mathcal{D}$.

(c) An object $X \in \mathcal{C}$ is called *R-local* if for all R-local equivalences $f\colon A \to B$ the map

$$f^*\colon \operatorname{Hom}_{\mathcal{C}}(B, X) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(A, X)$$

is bijective.

Example 9.2. (1) Let X be a topological space and consider the (full) inclusion $\mathrm{Shv}(X) \hookrightarrow \mathrm{PSh}(X)$ of sheaves into all presheaves. Its left adjoint, called sheafification is a Bousfield localization. The local equivalences are given by all maps $\mathcal{F} \to \mathcal{G}$ of presheaves such that $\mathcal{F}_x \xrightarrow{\sim} \mathcal{G}_x$ is an isomorphism for all $x \in X$.

(2) Consider the (full) inclusion $\mathsf{Ab} \hookrightarrow \mathsf{Grp}$ of abelian groups into all groups. Its left adjoint $G \mapsto G^{\mathrm{ab}}$, called *abelianization* is a Bousfield localization.

(3) Consider the (full) inclusion $\mathrm{Ban}_{\mathbb{R}} \hookrightarrow \mathrm{Norm}_{\mathbb{R}}$ of Banach spaces into the category of all normed real vector spaces with bounded linear operators. Its left adjoint, which is given by completion, is a Bousfield localization.

(4) (Later, see Theorem 15.6) Let R be a ring. Then the localization functor $Q\colon \mathsf{K}^+(R) \to \mathsf{D}^+(R)$ admits a fully faithful right adjoint given by taking injective resolutions.

Remark 9.3. Observe that the class of R-local equivalences has the following properties:

(a) Every isomorphism is an R-local equivalence.

(b) R-local equivalences satisfy the *2-out-of-3 property*: If f, g are composable morphisms in $\mathcal{C}$ and two of f, g, $g \circ f$ are isomorphisms, then so is the third.

Theorem 9.4. *Let $Q\colon \mathcal{C} \to \mathcal{D}$ be a Bousfield localization with fully faithful right adjoint R and unit $\eta\colon \mathrm{id}_{\mathcal{C}} \to RQ$.*

(i) For every $A \in \mathcal{C}$ the unit $\eta_A\colon A \to RQ(A)$ is an R-local equivalence.

(ii) Let $f\colon A \to B$ be an R-local equivalence between R-local objects. Then f is an isomorphism.

(iii) For every $D \in \mathcal{D}$ the object $R(D)$ is R-local.

(iv) Let $\mathcal{C}^{R\text{-loc}} \subseteq \mathcal{C}$ be the full subcategory of R-local objects. Then Q restricts to an equivalence of categories

$$Q'\colon \mathcal{C}^{R\text{-loc}} \xrightarrow{\sim} \mathcal{D}.$$

In particular, an object of $\mathcal{C}$ is R-local if and only if it is isomorphic to an object of the form $R(D)$ $(D \in \mathcal{D})$.

(v) The functor Q exhibits $\mathcal{D}$ as the localization of $\mathcal{C}$ at the class S of R-local equivalences:

$$\mathcal{D} = \mathcal{C}[S^{-1}].$$

In particular, $\mathcal{C}[S^{-1}]$ is a locally small category.

Proof. Recall that R being fully faithful means that the counit $\varepsilon\colon QR \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ is an isomorphism.

We prove (i), so let $A \in \mathcal{C}$. By the left triangle identity for adjunctions, we have $\varepsilon_{Q(A)} \circ Q(\eta_A) = \mathrm{id}_{Q(A)}$. As $\varepsilon_{Q(A)}$ and $\mathrm{id}_{Q(A)}$ are isomorphisms, so is $Q(\eta_A)$.

Let us prove (ii), so let $f\colon A \to B$ be an R-local equivalence between R-local objects. Since A is local, there exists a map $g\colon B \to A$ such that $g \circ f = \mathrm{id}_A$. By the 2-out-of-3 property (Remark 9.3), g is an R-local equivalence. As B is R-local, there exists a map $h\colon A \to B$ such that $h \circ g = \mathrm{id}_B$. It follows that g is an isomorphism, hence so is f.

For part (iii), let $D \in \mathcal{D}$ and let $f\colon A \to B$ be an R-local equivalence in $\mathcal{C}$. Since Q is left adjoint to R, we have a commutative diagram

$$\begin{tikzcd} \mathrm{Hom}_{\mathcal{C}}(B, R(D)) \arrow[r, "f^*"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathcal{C}}(A, R(D)) \arrow[d, "\sim"] \\ \mathrm{Hom}_{\mathcal{D}}(Q(B), D) \arrow[r, "\sim", "Q(f)"'] & \mathrm{Hom}_{\mathcal{D}}(Q(A), D), \end{tikzcd}$$

where the bottom horizontal map is an isomorphism because f is an R-local equivalence. We deduce that the top horizontal map is an isomorphism. As f was arbitrary, it follows that $R(D)$ is R-local.

We prove (iv). By part (iii) it is clear that Q' is left adjoint to R and that the counit $Q'R \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ is an isomorphism. It remains to prove that the unit $\eta\colon \mathrm{id}_{\mathcal{C}^{R\text{-loc}}} \to RQ'$ is an isomorphism. By (i) it suffices to show that every R-local equivalence $f\colon A \to B$ between R-local objects is invertible. But this is the content of (ii).

We finally prove (v). It is obvious from the definition that Q sends R-local equivalences to isomorphisms in $\mathcal{D}$. Let now $\mathcal{E}$ be another category. Observe that the adjunction $(Q, R, \eta, \varepsilon)$ induces an adjunction

$$R^*\colon \mathrm{Fun}(\mathcal{C}, \mathcal{E}) \rightleftarrows \mathrm{Fun}(\mathcal{D}, \mathcal{E}) : Q^*$$

with unit $\eta^*\colon \mathrm{id} \to Q^*R^*$ given by $\eta^*_F\colon F \xrightarrow{F\eta} FRQ = Q^*R^*(F)$ and counit $\varepsilon^*\colon R^*Q^* \xrightarrow{\sim} \mathrm{id}$ given by $\varepsilon^*_G\colon R^*Q^*(G) = GQR \xrightarrow{G\varepsilon} G$. We need to check the triangle identities $\varepsilon^*_{R^*F} \circ R^*\eta^*_F = \mathrm{id}_{R^*F}$ and $Q^*\varepsilon^*_G \circ \eta^*_{Q^*G} = \mathrm{id}_{Q^*G}$ for all functors $F\colon \mathcal{C} \to \mathcal{E}$ and $G\colon \mathcal{D} \to \mathcal{E}$. But this is equivalent to checking that the diagrams

$$\begin{tikzcd} FR \arrow[r, "F\eta R"] \arrow[dr, "\mathrm{id}_{FR}"'] & FRQR \arrow[d, "FR\varepsilon"] \\ & FR \end{tikzcd} \qquad \begin{tikzcd} GQ \arrow[r, "GQ\eta"] \arrow[dr, "\mathrm{id}_{GQ}"'] & GQRQ \arrow[d, "G\varepsilon Q"] \\ & GQ \end{tikzcd}$$

commute, which is clear from the triangle identities for $Q \dashv R$.

Since the counit is an isomorphism, we deduce that Q^* is fully faithful. The essential image of Q^* consists of all functors $F\colon \mathcal{C} \to \mathcal{E}$ sending R-local equivalences to isomorphisms in $\mathcal{E}$. Indeed, if F inverts R-local equivalences, then by (i) the natural transformation $F\eta\colon F \to FRQ = Q^*R^*F$ is an isomorphism, hence F lies in the essential image of Q^*. The other direction is trivial, because Q^* inverts R-local equivalences. $\square$

Remark 9.5. We will prove in Proposition 13.9 a criterion to determine when a localization functor is a Bousfield localization.

Exercise 9.6. Let $Q\colon \mathcal{C} \to \mathcal{D}$ be a Bousfield localization with fully faithful right adjoint R and let $\mathcal{I}$ be a small category. Show that a diagram $F\colon \mathcal{I} \to \mathcal{D}$ admits a (co)limit in $\mathcal{D}$ whenever $RF\colon \mathcal{I} \to \mathcal{C}$ admits one, and in that case the canonical maps

$$\operatorname{colim} F \xrightarrow{\sim} Q(\operatorname{colim} RF) \qquad \text{and} \qquad Q(\lim RF) \xrightarrow{\sim} \lim F$$

are isomorphisms in $\mathcal{D}$. Deduce that, if $\mathcal{C}$ is (co)complete, then so is $\mathcal{D}$.

Exercise 9.7. Let $Q\colon \mathcal{A} \to \mathcal{B}$ be a Bousfield localization between abelian categories, and suppose that Q is exact. Denote by R the fully faithful right adjoint of Q.

(i) Show that a morphism f in $\mathcal{A}$ is R-local if and only if $Q(\mathrm{Ker}(f)) = 0 = Q(\mathrm{Coker}(f))$.

(ii) Let $\mathcal{I}$ be a small category and suppose that the formation of $\mathcal{I}$-indexed (co)limits in $\mathcal{A}$ is exact. Show that $\mathrm{Ker}(Q)$ is closed under $\mathcal{I}$-indexed (co)limits if and only if Q preserves them.

(*Hint:* To show that Q preserves $\mathcal{I}$-indexed limits if $\mathrm{Ker}(Q)$ is closed under these, proceed as follows: Let $\mathcal{S}$ be the class of R-local equivalences in $\mathcal{A}$ and $\mathcal{T}$ the class of pointwise R-local equivalences in $\mathrm{Fun}(\mathcal{I}, \mathcal{A})$. Observe that $Q_*\colon\ \mathrm{Fun}(\mathcal{I}, \mathcal{A}) \rightleftarrows \mathrm{Fun}(\mathcal{I}, \mathcal{B}) : R_*$ is a Bousfield localization and $\mathcal{T}$ coincides with the R_*-local equivalences. The adjunction $\Delta\colon \mathcal{A} \rightleftarrows \mathrm{Fun}(\mathcal{I}, \mathcal{A}) : \lim$ satisfies $\Delta(\mathcal{S}) \subseteq \mathcal{T}$ and $\lim(\mathcal{T}) \subseteq \mathcal{S}$. Now conclude by Lemma 8.8.)

§10. Calculus of fractions

We now prove the existence of a localization in case we have a *calculus of fractions*.

Definition 10.1. Let $\mathcal{C}$ be a category. A class S of morphisms in $\mathcal{C}$ is called *(left) multiplicative* if it satisfies the following conditions:

(S1) S contains every isomorphism in $\mathcal{C}$.

(S2) If s, t are composable morphisms in S, then $s \circ t \in S$.

(S3) Every diagram $X \xleftarrow{f} Y \xrightarrow{s} Y'$ can be completed to a commutative diagram

$$\begin{tikzcd} Y \arrow[r, "s"] \arrow[d, "f"'] & Y' \arrow[d, dashed, "g"] \\ X \arrow[r, dashed, "t"'] & X' \end{tikzcd}$$

in $\mathcal{C}$ with $t \in S$.

(S4) Let $f, g\colon X \rightrightarrows Y$ be two morphisms in $\mathcal{C}$ and let $s\colon X' \to X$ in S with $f \circ s = g \circ s$. Then there exists $t\colon Y \to Y'$ in S such that $t \circ f = t \circ g$:

$$\begin{tikzcd} X' \arrow[r, "s"] & X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, dashed, "t"] & Y'. \end{tikzcd}$$

If these conditions are satisfied, we say that $\mathcal{C}$ admits a *(left) calculus of fractions* with respect to S.

Similarly, S is called a *right multiplicative system* if the corresponding class of morphisms in $\mathcal{C}^{\mathrm{op}}$ is a left multiplicative system. In this case, we say that $\mathcal{C}$ admits a right calculus of fractions.

Exercise 10.2. Show that Bousfield localizations admit a calculus of fractions with respect to the local equivalences.

Notation 10.3. Let $\mathcal{C}$ be a category which admits a calculus of fractions with respect to S, and fix $X \in \mathcal{C}$. Recall that the slice category $\mathcal{C}_{X/}$ is defined as the category whose objects are morphisms

$f\colon X \to Y$, and

$$\operatorname{Hom}_{\mathcal{C}_{X/}}\big(X \xrightarrow{f} Y, X \xrightarrow{g} Y'\big) \coloneqq \left\{ h\colon Y \to Y' \text{ in } \mathcal{C} \;\middle|\; \begin{tikzcd} & X \arrow[dl, "f"'] \arrow[dr, "g"] & \\ Y \arrow[rr, "h"'] & & Y' \end{tikzcd} \text{ commutes} \right\}.$$

We define $S^X \subseteq \mathcal{C}_{X/}$ as the full subcategory of all morphisms $s\colon X \to X'$ which lie in S. Beware that a morphism in S^X *need not* lie in S.

Lemma 10.4. *Suppose that $\mathcal{C}$ admits a calculus of fractions with respect to S and fix $X \in \mathcal{C}$. Then S^X is a filtered category.*

Proof. We need to check the following conditions:

(a) For any $s, s' \in S^X$ there exists $t \in S^X$ and morphisms $s \to t$, $s' \to t$.

(b) Let $f, g\colon s \rightrightarrows t$ be two morphisms in S^X. Then there exists $h\colon t \to u$ in S^X such that $h \circ f = h \circ g$.

Part (a) follows immediately from (S3) and (S2). Part (b) is a reformulation of (S4). □

Theorem 10.5. *Suppose that $\mathcal{C}$ admits a calculus of fractions with respect to S. Then the localization $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ exists. Moreover, Q commutes with finite colimites (provided these exist in $\mathcal{C}$).*

Proof. We define a category $\mathcal{C}_S$ via

$$\operatorname{Ob}(\mathcal{C}_S) \coloneqq \operatorname{Ob}(\mathcal{C}),$$

and, for all $X, Y \in \operatorname{Ob}(\mathcal{C}_S)$,

$$\operatorname{Hom}_{\mathcal{C}_S}(X, Y) \coloneqq \varinjlim_{[Y \to Y'] \in S^Y} \operatorname{Hom}_{\mathcal{C}}(X, Y').$$

By Lemma 10.4 the category S^Y is filtered. The elements of $\operatorname{Hom}_{\mathcal{C}_S}(X, Y)$ are equivalence classes $[f, s]$ represented by a *roof* (f, s) with $s \in S$, which we depict as

$$\begin{tikzcd} & Z & \\ X \arrow[ur, "f"] & & Y. \arrow[ul, "s"'] \end{tikzcd}$$

Two roofs $X \xrightarrow{f} Z \xleftarrow{s} Y$ and $X \xrightarrow{f'} Z' \xleftarrow{s'} Y$ are called equivalent, written $(f, s) \sim (f', s')$, if there exist morphisms $g\colon s \to s''$ and $g'\colon s' \to s''$ in S^Y such that $gf = g'f'$, that is, the diagram

$$\begin{tikzcd} & Z \arrow[d, "g"] & \\ X \arrow[ur, "f"] \arrow[r, "f''"] \arrow[dr, "f'"'] & Z'' & Y \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\ & Z' \arrow[u, "g'"'] & \end{tikzcd}$$

commutes.

Exercise. Show that $\sim$ is an equivalence relation.

We still need to define the composition in $\mathcal{C}_S$, for which we need some preparation.

Step 1: For every $Y \in \mathcal{C}_S$ and every morphism $h\colon [X \xrightarrow{s} X'] \to [X \xrightarrow{s'} X'']$ in S^X the map

$$h^*\colon \operatorname{Hom}_{\mathcal{C}_S}(X'', Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}_S}(X', Y)$$

is bijective. Note that $s^* h^* = s'^*$ as maps $\operatorname{Hom}_{\mathcal{C}_S}(X'', Y) \to \operatorname{Hom}_{\mathcal{C}_S}(X, Y)$. By the 2-out-of-3 property for isomorphisms, it suffices to show that s^* and s'^* are bijective. Without loss of generality it suffices to check this only for s^*. If $[f, t] \in \operatorname{Hom}_{\mathcal{C}_S}(X, Y)$ is arbitrary, then we can complete the diagram $Y' \xleftarrow{f} X \xrightarrow{s} X'$ by (S3) to a commutative diagram

$$\begin{tikzcd}
 & X \arrow[r, "s"] \arrow[d, "f"'] & X' \arrow[d, dashed, "f'"] \\
Y \arrow[r, "t"'] & Y' \arrow[r, dashed, "s'"'] & Y''
\end{tikzcd}$$

such that $s' \in S$. Then $s't \in S$ by (S2) and hence $s^*([f', s't]) = [f's, s't] = [s'f, s't] = [f, t]$ showing that s^* is surjective. In order to prove that s^* is injective, consider morphisms $f\colon X' \to Y'$, $t\colon Y \to Y'$, $f'\colon X' \to Y''$, $t'\colon Y \to Y''$ with $t, t' \in S^Y$ such that $s^*[f, t] = s^*[f', t']$. This means $(fs, t) \sim (f's, t')$, and hence we find maps $g\colon t \to t''$ and $g'\colon t' \to t''$ in S^Y such that $gfs = g'f's$. By (S4) there exists $h\colon t'' \to t'''$ in S^Y such that $hgf = hg'f'$. But now $(f, t) \sim (hgf, t''') = (hg'f', t''') \sim (f', t')$ showing that s^* is injective.

Step 2: For each map $h\colon [Y \xrightarrow{s} Y'] \to [Y \xrightarrow{s'} Y'']$ in S^Y, composition in $\mathcal{C}$ induces a map

$$\varinjlim_{[Y \to Y'] \in S^Y} \operatorname{Hom}_{\mathcal{C}_S}(Y', Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y') \to \operatorname{Hom}_{\mathcal{C}_S}(X, Z). \tag{10.1}$$

To see this, we have to check that the following diagram commutes:

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{C}_S}(Y'', Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y'') \arrow[r, "\circ"] \arrow[d, "h^* \times \mathrm{id}"', "\sim"] & \operatorname{Hom}_{\mathcal{C}_S}(X, Z) \\
\operatorname{Hom}_{\mathcal{C}_S}(Y', Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y'') & \operatorname{Hom}_{\mathcal{C}_S}(Y', Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y'). \arrow[l, "\mathrm{id} \times h_*"] \arrow[u, "\circ"']
\end{tikzcd} \tag{10.2}$$

To this end, take $f \in \operatorname{Hom}_{\mathcal{C}}(X, Y')$ and $[g, t] \in \operatorname{Hom}_{\mathcal{C}_S}(Y', Z)$ which is represented by a roof $Y' \xrightarrow{g} Z' \xleftarrow{t} Z$. Then

$$(\mathrm{id} \times h_*)([g, t], f) = \big([g, t], hf\big) \in \operatorname{Hom}_{\mathcal{C}_S}(Y', Z) \times \operatorname{Hom}_{\mathcal{C}}(X, Y'').$$

We now construct the preimage under $h^* \times \mathrm{id}$. From (S3) we get a commutative diagram

$$\begin{tikzcd}
Y \arrow[r, "gs"] \arrow[d, "hs = s'"'] & Z' \arrow[d, dashed, "s''"] \\
Y'' \arrow[r, "g'"'] & W,
\end{tikzcd}$$

where $s'' \in S$. This means $s''gs = g'hs$, and hence by (S4), there exists a map $u\colon W \to W'$ in S such that $us''g = ug'h$ as maps $Y' \to W'$. Hence $h^*([ug', us''t]) = [ug'h, us''t] = [us''g, us''t] = [g,t]$. Consequently, we get $(h^* \times \mathrm{id})([ug', us''t], hf) = ([g,t], hf) = (\mathrm{id} \times h_*)([g,t], f)$. We compute

$$[g,t] \circ f = [gf, t] = [ug'hf, us''t] = [ug', us''t] \circ hf,$$

which shows that the diagram (10.2) commutes.

Step 3: $\mathcal{C}_S$ is a category. We define the composition on $\mathcal{C}_S$ via the map

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_S}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}_S}(X,Y) &= \mathrm{Hom}_{\mathcal{C}_S}(Y,Z) \times \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}}(X,Y') \\
&= \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}_S}(Y,Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y') \\
&\xleftarrow{\sim} \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}_S}(Y',Z) \times \mathrm{Hom}_{\mathcal{C}}(X,Y') && \text{(Step 1)} \\
&\longrightarrow \mathrm{Hom}_{\mathcal{C}_S}(X,Z) && \text{(by (10.1))}.
\end{aligned}$$

Concretely, let $X \xrightarrow{f} Y' \xleftarrow{s} Y$ and $Y \xrightarrow{g} Z' \xleftarrow{t} Z$ be two roofs with $s, t \in S$. The composite $[g,t] \circ [f,s]$ is then given by the roof $[g'f, s't]$,

$$\begin{tikzcd}
 & & Z'' & & \\
 & Y' \arrow[ur, dashed, "g'"] & & Z' \arrow[ul, dashed, "s'"'] & \\
X \arrow[ur, "f"] & & Y \arrow[ul, "s"'] \arrow[ur, "g"] & & Z, \arrow[ul, "t"']
\end{tikzcd}$$

where g', s' are morphisms such that $g's = s'g$ and $s' \in S$ (which exist by (S3)); note that $s't \in S$ by (S2). The construction above shows that this definition is independent of the choices of g', s' and of the chosen representatives of $[g,t]$ and $[f,s]$.

It is obvious that the identity id_X is represented by the roof $(\mathrm{id}_X, \mathrm{id}_X)$ (where $\mathrm{id}_X \in S$ by (S1)). The associativity is trivial to check and left as an exercise.

Step 4: Construction of a canonical functor $Q\colon \mathcal{C} \to \mathcal{C}_S$ which inverts S. Define Q as the identity on objects and as the canonical inclusion

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X,Y) &\longrightarrow \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}}(X,Y') = \mathrm{Hom}_{\mathcal{C}_S}(X,Y), \\
f &\longmapsto [f, \mathrm{id}_Y]
\end{aligned}$$

corresponding to $\mathrm{id}_Y \in S$. We now check that Q inverts S: Let $s\colon X \to X'$ be in S, viewed as an object of S^X. Then precomposition with $Q(s) = [s, \mathrm{id}_{X'}]$ is given by the isomorphism in Step 1. Hence $Q(s)$ is an isomorphism by the Yoneda lemma.

Step 5: $Q\colon \mathcal{C} \to \mathcal{C}_S$ is a localization functor. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor which inverts S. We define a functor $\overline{F}\colon \mathcal{C}_S \to \mathcal{D}$ by F on objects and on morphisms by

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}_S}(X,Y) = \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}}(X,Y') &\xrightarrow{F} \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y')) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(F(X), F(Y)),
\end{aligned}$$

where the equivalence comes from the fact that the transition maps are bijective, because F inverts all morphisms in S^Y by the 2-out-of-3 property. It is clear that $\overline{F}$ indeed defines a functor and satisfies $F = \overline{F} \circ Q$ (strict equality!). It remains to show that, for two functors $\overline{F}, \overline{G} \colon \mathcal{C}_S \rightrightarrows \mathcal{D}$ the natural map

$$\begin{aligned} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C}_S,\mathcal{D})}(\overline{F},\overline{G}) &\xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{D})}(\overline{F}Q,\overline{G}Q), \\ \alpha &\longmapsto \alpha Q \end{aligned} \tag{10.3}$$

is bijective. Injectivity is obvious from the fact that Q is the identity on objects. For surjectivity, let $\beta \colon \overline{F}Q \to \overline{G}Q$ be a natural transformation; hence for each $X \in \mathcal{C}$ we are given a morphism $\beta_X \colon \overline{F}Q(X) \to \overline{G}Q(X)$ in $\mathcal{D}$ such that for all maps $f \colon X \to Y$ the diagram

$$\begin{tikzcd} \overline{F}Q(X) \arrow[r, "\beta_X"] \arrow[d, "\overline{F}Q(f)"'] & \overline{G}Q(X) \arrow[d, "\overline{G}Q(f)"] \\ \overline{F}Q(Y) \arrow[r, "\beta_Y"'] & \overline{G}Q(Y) \end{tikzcd}$$

commutes in $\mathcal{D}$. For each $X \in \mathrm{Ob}(\mathcal{C}_S)$ we put $\alpha_X = \beta_X$. For each morphism $\overline{f} \colon X \to Y$ in $\mathcal{C}_S$, represented by a roof $X \xrightarrow{f} Y' \xleftarrow{s} Y$, we have $\overline{f} = Q(s)^{-1} \circ Q(f)$ and hence a commutative diagram

$$\begin{tikzcd} \overline{F}(X) \arrow[rrr, bend left, "\alpha_X"] \arrow[r, equal] \arrow[dd, "\overline{F}(\overline{f})"'] & \overline{F}Q(X) \arrow[r, "\beta_X"] \arrow[d, "\overline{F}Q(f)"'] & \overline{G}Q(X) \arrow[r, equal] \arrow[d, "\overline{G}Q(f)"] & \overline{G}(X) \arrow[dd, "\overline{G}(\overline{f})"] \\ & \overline{F}Q(Y') \arrow[r, "\beta_{Y'}"] & \overline{G}Q(Y') & \\ \overline{F}(Y) \arrow[rrr, bend right, "\alpha_Y"'] \arrow[r, equal] & \overline{F}Q(Y) \arrow[r, "\beta_Y"'] \arrow[u, "\overline{F}Q(s)", "\sim"'] & \overline{G}Q(Y) \arrow[r, equal] \arrow[u, "\sim", "\overline{G}Q(s)"'] & \overline{G}(Y), \end{tikzcd}$$

which shows $\beta = \alpha Q$. Therefore the map (10.3) is bijective finishing the proof that $(\mathcal{C}_S, Q)$ satisfies the universal property of a localization.

It remains to prove that $Q \colon \mathcal{C} \to \mathcal{C}_S$ preserves finite colimits. To this end, let $\mathcal{I}$ be a finite category and $F \colon \mathcal{I} \to \mathcal{C}$ a diagram. For any $Y \in \mathcal{C}_S$ we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}_{\mathcal{C}_S}\big(Q(\operatorname*{colim}_{\mathcal{I}} F), Y\big) &\cong \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}}\big(\operatorname*{colim}_{\mathcal{I}} F, Y'\big) \\ &\cong \varinjlim_{[Y \to Y'] \in S^Y} \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}\big(F(i), Y'\big) \\ &\cong \lim_{i \in \mathcal{I}^{\mathrm{op}}} \varinjlim_{[Y \to Y'] \in S^Y} \mathrm{Hom}_{\mathcal{C}}\big(F(i), Y'\big) \\ &\cong \lim_{i \in \mathcal{I}^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}_S}\big(QF(i), Y\big), \end{aligned}$$

where the third isomorphism uses the well-known fact that filtered colimits in Set commute with finite limits (see, *e.g.*, [KS06, Theorem 3.1.6]). This shows that $Q \operatorname{colim}_{\mathcal{I}} F$ is the colimit of $QF \colon \mathcal{I} \to \mathcal{C}_S$. $\square$

We draw the following immediate consequences from the proof of Theorem 10.5.

Corollary 10.6. *Suppose that $\mathcal{C}$ admits a calculus of fractions with respect to S, and denote $Q\colon \mathcal{C} \to \mathcal{C}_S$ the localization functor.*

(i) Every morphism in $\mathcal{C}_S$ is of the form $Q(s)^{-1} \circ Q(f)$ with $s \in S$.

(ii) Let $f, g\colon X \rightrightarrows Y$ be two morphisms in $\mathcal{C}$. Then $Q(f) = Q(g)$ if and only if there exists $s\colon Y \to Z$ in S such that $s \circ f = s \circ g$.

(iii) Let $f\colon X \to Y$ be a morphism in $\mathcal{C}$. Then $Q(f)$ is an isomorphism if and only if there exist morphisms $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f \in S$ and $h \circ g \in S$.

Proof. For (i), we observe that every morphism in $\mathcal{C}_S$ is represented by a roof (f, s) and that $Q(s) \circ [f, s] = Q(f)$ by (the proof of) Theorem 10.5.

We now prove (ii). By definition, we have $Q(f) = Q(g)$ if and only if there exist $s, t\colon Y \to Z$ in S such that the diagram

$$\begin{tikzcd}
& Y \arrow[d, "s"] \arrow[dr, equal] & \\
X \arrow[ur, "f"] \arrow[dr, "g"'] & Z & Y \\
& Y \arrow[u, "t"'] \arrow[ur, equal] &
\end{tikzcd}$$

commutes. But this is equivalent to $s \circ f = t \circ g$ and $s = t$.

Finally we show (iii). For the "only if" direction it suffices to show that if $Q(f)$ is an isomorphism, then there exists a morphism g with $g \circ f \in S$. Because then $Q(g) \circ Q(f) = Q(g \circ f)$ and $Q(f)$ are isomorphisms, hence so is $Q(g)$. But then there exists a morphism h such that $h \circ g \in S$.

Suppose now that $Q(f)$ is an isomorphism. The inverse is represented by a roof $Y \xrightarrow{g'} Z' \xleftarrow{s} X$, from which we deduce $Q(g' \circ f) = Q(g') \circ Q(f) = Q(s)$. By (ii) there exists $t\colon Z' \to Z$ in S such that $t \circ g' \circ f = t \circ s \in S$. Hence $g \coloneqq t \circ g'$ is as desired.

Conversely, suppose that there are morphisms $g\colon Y \to Z$ and $h\colon Z \to W$ such that $g \circ f \in S$ and $h \circ g \in S$. Then $Q(g) \circ Q(f) = Q(g \circ f)$ and $Q(h) \circ Q(g) = Q(h \circ g)$ are isomorphisms. We deduce that $Q(g)$ is an isomorphism, and then $Q(f)$ is one as well by 2-out-of-3. $\square$

Remark 10.7. Suppose that $\mathcal{C}$ admits a calculus of fractions with respect to S. The proof of Theorem 10.5 shows that the localization $\mathcal{C}_S$ is locally small if and only if for all $X \in \mathcal{C}$ the filtered category S^X is cofinally small.

Proposition 10.8. *Suppose that $\mathcal{C}$ admits a calculus of fractions with respect to S. Let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory such that either the conditions*

(a) $S \cap \mathcal{D}$ (i.e., the class of morphisms $X \to Y$ in S with $X, Y \in \mathcal{D}$) is a multiplicative system;

(b) For all $s\colon X \to X'$ in S with $X \in \mathcal{D}$ there exists $g\colon X' \to Y$ with $g \circ s \in S \cap \mathcal{D}$;

or the conditions

(a') $S \cap \mathcal{D}$ is a right multiplicative system;

(b') For all $s\colon Y' \to Y$ in S with $Y \in \mathcal{D}$ there exists $g\colon X \to Y'$ with $s \circ g \in S \cap \mathcal{D}$;

are satisfied. Then the induced functor $\mathcal{D}_{S\cap\mathcal{D}} \hookrightarrow \mathcal{C}_S$ is fully faithful.

Proof. By (a) and Theorem 10.5 the localization $Q_{\mathcal{D}}\colon \mathcal{D} \to \mathcal{D}_{S\cap\mathcal{D}}$ exists. Since the composite $\mathcal{D} \hookrightarrow \mathcal{C} \xrightarrow{Q_{\mathcal{C}}} \mathcal{C}_S$ inverts the morphisms in $S \cap \mathcal{D}$, there is a canonical functor $\mathcal{D}_{S\cap\mathcal{D}} \to \mathcal{C}_S$ (which is compatible with the localization functors). We need to check that for all $X, Y \in \mathcal{D}$ the induced map

$$\mathrm{Hom}_{\mathcal{D}_{S\cap\mathcal{D}}}(X,Y) \longrightarrow \mathrm{Hom}_{\mathcal{C}_S}(X,Y)$$

is bijective. Let $f\colon X \to Y$ be a morphism in $\mathcal{C}_S$, which is represented by some roof $X \xrightarrow{g} Y' \xleftarrow{s} Y$. By (b) there exists a map $t\colon Y' \to Z$ such that $ts \in S \cap \mathcal{D}$. Then f is also represented by the roof $X \xrightarrow{tg} Z \xleftarrow{ts} Y$ in $\mathcal{D}$, so that f lies in the image of $\mathcal{D}_{S\cap\mathcal{D}}$ proving surjectivity. In order to prove injectivity, let $f, f'\colon X \to Y$ be morphisms in $\mathcal{D}_{S\cap\mathcal{D}}$ represented by roofs $X \xrightarrow{\overline{f}} Z \xleftarrow{s} Y$ and $X \xrightarrow{\overline{f}'} Z' \xleftarrow{s'} Y$, respectively. Supposing $f = f'$ in $\mathcal{C}_S$, there exist morphisms $u\colon s \to s''$ and $u'\colon s' \to s''$ in S^Y such that the diagram

$$\begin{tikzcd}
 & Z \arrow[d, "u"] & \\
X \arrow[ur, "\overline{f}"] \arrow[r, "\overline{f}''"] \arrow[dr, "\overline{f}'"'] & Z'' & Y \arrow[ul, "s"'] \arrow[l, "s''"'] \arrow[dl, "s'"] \\
 & Z' \arrow[u, "u'"'] &
\end{tikzcd}$$

commutes in $\mathcal{C}$. By (b) again, there exists $t\colon Z'' \to W$ such that $ts'' \in S \cap \mathcal{D}$. Then f and f' are both represented by the roofs $(t\overline{f}'', ts'')$, which proves injectivity. $\square$

Corollary 10.9. *Suppose $\mathcal{C}$ admits a calculus of fractions with respect to S, and let $\mathcal{D} \subseteq \mathcal{C}$ be a full subcategory satisfying the following condition:*

($*$) For all $X \in \mathcal{C}$ there exists $s\colon X \to Y$ in S with $Y \in \mathcal{D}$.

Then $S \cap \mathcal{D}$ is a multiplicative system and the functor $\mathcal{D}_{S\cap\mathcal{D}} \xrightarrow{\simeq} \mathcal{C}_S$ is an equivalence.

Proof. It is clear that $S \cap \mathcal{D}$ satisfies (S1) and (S2), and (S3), (S4) follow immediately from ($*$) and the fact that S is a multiplicative system. Hence $S \cap \mathcal{D}$ is a multiplicative system. Moreover, ($*$) also implies condition (b) in Proposition 10.8, hence we deduce that $\mathcal{D}_{S\cap\mathcal{D}} \hookrightarrow \mathcal{C}_S$ is fully faithful. It is essentially surjective again by ($*$). $\square$

§11. Localization of triangulated categories

Definition 11.1. Let $(\mathcal{C}, T)$ be a triangulated category and S a class of morphisms satisfying (S1)–(S4). We say that S is *compatible with the triangulation* if it satisfies the following conditions:

(S5) $s \in S \iff T(s) \in S$.

(S6) Consider a commutative diagram with solid arrows

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[r, "h"] \arrow[d, dashed, "u"] & T(X) \arrow[d, "T(s)"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X'), \end{tikzcd}$$

where the rows are distinguished triangles, and suppose that $s, t \in S$. Then there exists $u \in S$ making the whole diagram commute.

Theorem 11.2. *Let $(\mathcal{C}, T)$ be a triangulated category and S a multiplicative system which is compatible with the triangulation. Then:*

(i) There is a unique (up to unique isomorphism) pair (T_S, ξ) consisting of an equivalence $T_S \colon \mathcal{C}_S \xrightarrow{\sim} \mathcal{C}_S$ and a natural isomorphism $\xi \colon Q \circ T \xrightarrow{\sim} T_S \circ Q$.

(ii) $(\mathcal{C}_S, T_S)$ has the unique structure of a triangulated category such that the localization functor $(Q, \xi) \colon (\mathcal{C}, T) \to (\mathcal{C}_S, T_S)$ is exact.

(iii) The pair $(\mathcal{C}_S, Q)$ satisfies the following universal property: For all triangulated categories $(\mathcal{D}, T_{\mathcal{D}})$, precomposition with Q induces a fully faithful functor

$$Q^* \colon \operatorname{Fun}^{\triangle}(\mathcal{C}_S, \mathcal{D}) \hookrightarrow \operatorname{Fun}^{\triangle}(\mathcal{C}, \mathcal{D})$$

whose essential image consists of those exact functors which invert S.

Proof. We first argue that $\mathcal{C}_S$ is additive. Let $\overline{S}$ be the class of morphisms in $\mathcal{C}$ which are inverted by Q. Then the induced functor $\mathcal{C}_S \xrightarrow{\sim} \mathcal{C}_{\overline{S}}$ is an equivalence compatible with the localization functors. From Theorem 10.5 we deduce that $Q \colon \mathcal{C} \to \mathcal{C}_{\overline{S}}$ preserves finite coproducts. Hence, $\overline{S}$ is closed under coproducts (which are biproducts in $\mathcal{C}$). We may thus apply Proposition 8.10 which shows that $\mathcal{C}_{\overline{S}} \cong \mathcal{C}_S$ is additive.

For part (i) we note that the functors $T^{-1} \colon \mathcal{C} \rightleftarrows \mathcal{C} : T$ form an adjoint equivalence satisfying $T^{-1}(S) \subseteq S$ and $T(S) \subseteq S$ by (S5). Now, by Lemma 8.8 we obtain an induced adjoint equivalence $T_S^{-1} \colon \mathcal{C}_S \rightleftarrows \mathcal{C}_S : T_S$ and a natural isomorphism $\xi \colon Q \circ T \xrightarrow{\sim} T_S \circ Q$. Since Q is a localization, the pair (T_S, ξ) is unique (up to unique isomorphism) by Remark 8.3, which proves (i).

Let us prove (ii). We call a triangle in $\mathcal{C}_S$ distinguished if it is isomorphic to a triangle of the form

$$Q(X) \xrightarrow{Q(f)} Q(Y) \xrightarrow{Q(g)} Q(Z) \xrightarrow{\xi_X Q(h)} T_S Q(X) \tag{11.1}$$

for some triangle (f, g, h) in $\mathcal{C}$. Then clearly $(\mathrm{id}_{Q(X)}, 0, 0)$ is a triangle in $\mathcal{C}_S$. Let $X \to Y$ be a morphism in $\mathcal{C}_S$ represented by some roof $X \xrightarrow{f} Y' \xleftarrow{s} Y$. By (T1) for $\mathcal{C}$ there exists a triangle $X \xrightarrow{f} Y' \xrightarrow{g} Z \xrightarrow{h} T(X)$ in $\mathcal{C}$; but then we obtain an isomorphism of triangles

$$\begin{tikzcd} Q(X) \arrow[r, "Q(s)^{-1}Q(f)"] \arrow[d, equal] & Q(Y) \arrow[r, "Q(gs)"] \arrow[d, "Q(s)"] & Q(Z) \arrow[r, "\xi_X Q(h)"] \arrow[d, equal] & T_S Q(X) \arrow[d, equal] \\ Q(X) \arrow[r, "Q(f)"'] & Q(Y') \arrow[r, "Q(g)"'] & Q(Z) \arrow[r, "\xi_X Q(h)"'] & T_S Q(X). \end{tikzcd}$$

Therefore, (T1) holds for $\mathcal{C}_S$.

We now verify (T2). Since the condition in (T2) is invariant under isomorphisms of triangles, we may check it for the standard distinguished triangles. So consider the distinguished triangle (11.1). We obtain an isomorphism of triangles

$$\begin{tikzcd}
Q(Y) \arrow[r, "Q(g)"] \arrow[d, equal] & Q(Z) \arrow[r, "Q(h)"] \arrow[d, equal] & Q(T(X)) \arrow[r, "-\xi_Y QT(f)"] \arrow[d, "\xi_X"] & T_S Q(Y) \arrow[d, equal] \\
Q(Y) \arrow[r, "Q(g)"'] & Q(Z) \arrow[r, "\xi_X Q(h)"'] & T_S Q(X) \arrow[r, "-T_S Q(f)"'] & T_S Q(Y),
\end{tikzcd}$$

where the top triangle is distinguished and the right rectangle commutes by naturality of ξ. Thus, (T2) is satisfied.

We now check (T3). Let $f\colon Q(X) \to Q(Y)$ and $g\colon Q(Y) \to Q(Z)$ be two morphisms in $\mathcal{C}_S$ and put $h = g \circ f$. Let f and g be represented by the roofs $X \xrightarrow{f_0} Y_0 \xleftarrow{s} Y$ and $Y \xrightarrow{g_1} Z_1 \xleftarrow{t} Z$, respectively. By (T1) we can embed f_0 into a distinguished triangle $X = X_0 \xrightarrow{f_0} Y_0 \xrightarrow{f_0'} Z_0' \xrightarrow{f_0''} T(X_0)$. Note that we have an isomorphism of triangles

$$\begin{tikzcd}
Q(X) \arrow[r, "f"] \arrow[d, equal] & Q(Y) \arrow[r, "Q(f_0's)"] \arrow[d, "Q(s)"] & Q(Z_0') \arrow[r, "\xi_X Q(f_0'')"] \arrow[d, equal] & T_S Q(X) \arrow[d, equal] \\
Q(X_0) \arrow[r, "Q(f_0)"'] & Q(Y_0) \arrow[r, "Q(f_0')"'] & Q(Z_0') \arrow[r, "\xi_{X_0} Q(f_0'')"'] & T_S Q(X_0).
\end{tikzcd}$$

By (S3) and (T1), the diagram $Y_0 \xleftarrow{s} Y \xrightarrow{g_1} Z_1$ can be completed to a commutative diagram

(11.2)
$$\begin{tikzcd}
Y \arrow[r, "g_1"] \arrow[d, "s"'] & Z_1 \arrow[r, "g_1'"] \arrow[d, "s_0"] & X_1' \arrow[r, "g_1''"] \arrow[d, dashed, "s_0'"] & T(Y) \arrow[d, "T(s)"] \\
Y_0 \arrow[r, "g_0"'] & Z_0 \arrow[r, "g_0'"'] & X_0' \arrow[r, "g_0''"'] & T(Y_0),
\end{tikzcd}$$

with $s_0 \in S$, and then (S6) provides $s_0' \in S$ making the whole diagram commutative. Hence, the image of (11.2) under Q is an isomorphism of triangles. Thus, we obtain an isomorphism of triangles

$$\begin{tikzcd}
Q(Y) \arrow[r, "g"] \arrow[d, "Q(s)"'] & Q(Z) \arrow[r, "Q(g_1't)"] \arrow[d, "Q(s_0t)"] & Q(X_1') \arrow[r, "\xi_Y Q(g_1'')"] \arrow[d, "Q(s_0')"] & T_S Q(Y) \arrow[d, "T_S Q(s)"] \\
Q(Y_0) \arrow[r, "Q(g_0)"'] & Q(Z_0) \arrow[r, "Q(g_0')"'] & Q(X_0') \arrow[r, "\xi_{Y_0} Q(g_0'')"'] & T_S Q(Y_0).
\end{tikzcd}$$

(Here, the right rectangle commutes, since $\xi_{Y_0} Q(g_0'') Q(s_0') = \xi_{Y_0} Q(g_0'' s_0') = \xi_{Y_0}(Q(T(s) g_1'')) = \xi_{Y_0} QT(s) Q(g_1'') = T_S Q(s) \xi_Y Q(g_1'')$ by naturality of ξ.)

Note that $h = Q(s_0 t)^{-1} Q(h_0)$, where $h_0 := g_0 f_0$. Hence, embedding h_0 in a distinguished triangle $X_0 \xrightarrow{h_0} Z_0 \xrightarrow{h'_0} Y'_0 \xrightarrow{h''_0} T(X_0)$, we obtain an isomorphism of triangles

$$\begin{tikzcd}
Q(X) \arrow[r, "h"] \arrow[d, equal] & Q(Z) \arrow[r, "Q(h'_0 s_0 t)"] \arrow[d, "Q(s_0 t)"] & Q(Y'_0) \arrow[r, "\xi_X Q(h''_0)"] \arrow[d, equal] & T_S Q(X) \arrow[d, equal] \\
Q(X_0) \arrow[r, "Q(h_0)"'] & Q(Z_0) \arrow[r, "Q(h'_0)"'] & Q(Y'_0) \arrow[r, "\xi_{X_0} Q(h''_0)"'] & T_S Q(X_0).
\end{tikzcd}$$

Applying (T3) for $\mathcal{C}$ to the morphisms $X_0 \xrightarrow{f_0} Y_0$ and $Y_0 \xrightarrow{g_0} Z_0$, we obtain a distinguished triangle

$$Z'_0 \xrightarrow{u_0} Y'_0 \xrightarrow{v_0} X'_0 \xrightarrow{T(f'_0)g''_0} T(Z'_0)$$

in $\mathcal{C}$ such that the following diagram commutes:

$$\begin{tikzcd}
 & & Z'_0 \arrow[rr, bend left, "f''_0"] \arrow[dr, dashed, "u_0"] & & T(X_0) \arrow[dr, "T(f_0)"] & & \\
 & Y_0 \arrow[ur, "f'_0"] \arrow[dr, "g_0"] & & Y'_0 \arrow[ur, "h''_0"] \arrow[dr, dashed, "v_0"] & & T(Y_0) \arrow[dr, "T(f')"] & \\
X_0 \arrow[ur, "f_0"] \arrow[rr, bend right, "h_0"'] & & Z_0 \arrow[ur, "h'_0"] \arrow[rr, bend right, "g'_0"'] & & X'_0 \arrow[ur, "g''_0"] \arrow[rr, bend right, "T(f'_0)g''_0"'] & & T(Y/X).
\end{tikzcd}$$

Now, put $u \coloneqq Q(u_0) \colon Q(Z'_0) \to Q(Y'_0)$ and $v \coloneqq Q(s'_0)^{-1} Q(v_0) \colon Q(Y'_0) \to Q(X'_1)$, so that we have an isomorphism of triangles

$$\begin{tikzcd}
Q(Z'_0) \arrow[r, "u"] \arrow[d, equal] & Q(Y'_0) \arrow[r, "v"] \arrow[d, equal] & Q(X'_1) \arrow[r, "T_S Q(f'_0 s) \circ \xi_Y Q(g''_1)"] \arrow[d, "Q(s'_0)"] & T_S Q(Z'_0) \arrow[d, equal] \\
Q(Z'_0) \arrow[r, "Q(u_0)"'] & Q(Y'_0) \arrow[r, "Q(v_0)"'] & Q(X'_0) \arrow[r, "T_S Q(f'_0) \circ \xi_{Y_0} Q(g''_0)"'] & T_S Q(Z'_0),
\end{tikzcd}$$

where the bottom triangle is distinguished by construction. We put

$$\begin{aligned}
f' &\coloneqq Q(f'_0 s), & f'' &\coloneqq \xi_X Q(f''_0), \\
g' &\coloneqq Q(g'_1 t), & g'' &\coloneqq \xi_Y Q(g''_1), \\
h' &\coloneqq Q(h'_0 s_0 t), & h'' &\coloneqq \xi_X Q(h''_0)
\end{aligned}$$

in order to simplify the notation. We then have to show that the following diagram commutes:

$$\begin{tikzcd}
 & & Q(Z_0') \arrow[rr, bend left, "f''"] \arrow[dr, dashed, "u"] & & T_S Q(X) \arrow[dr, "T_S(f)"] & & \\
 & Q(Y) \arrow[ur, "f'"] \arrow[dr, "g"] & & Q(Y_0') \arrow[ur, "h''"] \arrow[dr, dashed, "v"] & & T_S Q(Y) \arrow[dr, "T_S(f')"] & \\
Q(X) \arrow[ur, "f"] \arrow[rr, bend right, "h"'] & & Q(Z) \arrow[ur, "h'"] \arrow[rr, bend right, "g'"'] & & Q(X_1') \arrow[ur, "g''"] \arrow[rr, bend right, "T_S(f')g''"'] & & T_S Q(Z_0').
\end{tikzcd}$$

To this end, we compute

$$\begin{aligned}
uf' &= Q(u_0 f_0' s) = Q(h_0' g_0 s) = Q(h_0' s_0 g_1) = Q(h_0' s_0 t) \circ Q(t)^{-1} Q(g_1) = h' g, \\
h'' u &= \xi_X Q(h_0'' u_0) = \xi_X Q(f_0'') = f'', \\
vh' &= Q(s_0')^{-1} Q(v_0 h_0' s_0 t) = Q(s_0')^{-1} Q(g_0' s_0 t) = Q(s_0')^{-1} Q(s_0' g_1' t) = Q(g_1' t) = g', \\
T_S(f) h'' &= T_S Q(s)^{-1} T_S Q(f_0) \circ \xi_X Q(h_0'') = \xi_Y Q T(s)^{-1} Q\big(T(f_0) h_0''\big) \\
&= \xi_Y Q T(s)^{-1} Q(g_0'' v_0) = \xi_Y Q(g_1'') Q(s_0')^{-1} Q(v_0) = g'' v.
\end{aligned}$$

Hence $\mathcal{C}_S$ satisfies the axiom (T3). This finishes the proof that $\mathcal{C}_S$ is triangulated and that Q is exact. The uniqueness assertions are obvious from the construction.

Finally, we prove (iii). Let $(\mathcal{D}, T_{\mathcal{D}})$ be a triangulated category. By (ii) the functor (Q, ξ) is exact, hence by Exercise 6.2 we obtain a functor

$$Q^* \colon \operatorname{Fun}^{\triangle}(\mathcal{C}_S, \mathcal{D}) \longrightarrow \operatorname{Fun}^{\triangle}(\mathcal{C}, \mathcal{D}).$$

To finish the proof, we need to show the following claims:

(a) An exact functor $F \colon \mathcal{C} \to \mathcal{D}$ lies in the essential image of Q^* if and only if F inverts S.

(b) The map $\operatorname{Hom}_{\operatorname{Fun}^{\triangle}(\mathcal{C}_S, \mathcal{D})}(\overline{F}, \overline{G}) \to \operatorname{Hom}_{\operatorname{Fun}^{\triangle}(\mathcal{C}, \mathcal{D})}(\overline{F}Q, \overline{G}Q)$, $\mu \mapsto \mu Q$ is bijective for all exact functors $\overline{F}, \overline{G} \in \operatorname{Fun}^{\triangle}(\mathcal{C}_S, \mathcal{D})$.

It is clear that every functor in the essential image of Q^* inverts S. So let $(F, \zeta) \colon (\mathcal{C}, T) \to (\mathcal{D}, T_{\mathcal{D}})$ be an exact functor which inverts S. By the universal property of Q there exists a unique (up to unique isomorphism) pair $(\overline{F}, \mu)$ consisting of a functor $\overline{F} \colon \mathcal{C}_S \to \mathcal{D}$ and a natural isomorphism $\mu \colon F \xrightarrow{\sim} \overline{F} \circ Q$. Define a natural isomorphism $\overline{\zeta}' \colon \overline{F} T_S Q \xrightarrow{\sim} T_{\mathcal{D}} \overline{F} Q$ by the commutativity of the following diagram:

(11.3)
$$\begin{tikzcd}
FT \arrow[rr, "\zeta", "\sim"'] \arrow[d, "\mu T"', "\sim"] & & T_{\mathcal{D}} F \arrow[d, "\sim"', "T_{\mathcal{D}} \mu"] \\
\overline{F} Q T \arrow[r, "\sim", "\overline{F}\xi"'] & \overline{F} T_S Q \arrow[r, "\sim", "\overline{\zeta}'"'] & T_{\mathcal{D}} \overline{F} Q.
\end{tikzcd}$$

Since the map $Q^*\colon \operatorname{Hom}(\overline{F}T_S, S\overline{F}) \xrightarrow{\sim} \operatorname{Hom}(\overline{F}T_SQ, T_{\mathcal{D}}\overline{F}Q)$ is bijective, we find a unique natural isomorphism $\overline{\zeta}\colon \overline{F}T_S \xrightarrow{\sim} T_{\mathcal{D}}\overline{F}$ such that $\overline{\zeta}' = \overline{\zeta}Q$. Using (11.3), it is now trivial to check that $(\overline{F}, \overline{\zeta})$ preserves distinguished triangles. This shows (a).

We finally prove (b). Let $(\overline{F}, \overline{\zeta}), (\overline{G}, \overline{\theta})\colon \mathcal{C}_S \to \mathcal{D}$ be exact functors. Note that we have a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_{\operatorname{Fun}^{\triangle}(\mathcal{C}_S,\mathcal{D})}(\overline{F},\overline{G}) \arrow[r] \arrow[d, hook] & \operatorname{Hom}_{\operatorname{Fun}^{\triangle}(\mathcal{C},\mathcal{D})}(\overline{F}Q,\overline{G}Q) \arrow[d, hook] \\ \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}_S,\mathcal{D})}(\overline{F},\overline{G}) \arrow[r, "\sim"] & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{D})}(\overline{F}Q,\overline{G}Q), \end{tikzcd} \tag{11.4}$$

hence the top arrow is injective. Let now $\mu\colon \overline{F} \to \overline{G}$ be a natural transformation such that μQ is exact. Consider the following diagram of natural isomorphisms:

$$\begin{tikzcd} \overline{F}QT \arrow[r, "\overline{F}\xi"] \arrow[d, "\mu QT"'] & \overline{F}T_SQ \arrow[r, "\overline{\zeta}Q"] \arrow[d, "\mu T_S Q"'] & T_{\mathcal{D}}\overline{F}Q \arrow[d, "T_{\mathcal{D}}\mu Q"] \\ \overline{G}QT \arrow[r, "\overline{G}\xi"'] & \overline{G}T_SQ \arrow[r, "\overline{\theta}Q"'] & T_{\mathcal{D}}\overline{G}Q. \end{tikzcd}$$

Here, the outher rectangle commutes by assumption and the left square commutes by naturality. Hence, the right square commutes. Since the bottom map in (11.4) is bijective, we deduce that $T_{\mathcal{D}}\mu \circ \overline{\zeta} = \overline{\theta} \circ \mu T_S$. Hence μ is exact. □

Theorem 11.3. *Let $(\mathcal{C}, T)$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{C}$ a triangulated subcategory (Definition 6.6). Let $S_{\mathcal{N}}$ be the class of morphisms $f\colon X \to Y$ in $\mathcal{C}$ such that for every distinguished triangle $X \xrightarrow{f} Y \to N \to T(X)$ we have $N \in \mathcal{N}$.*

Then:

(i) $S_{\mathcal{N}}$ is a left and right multiplicative system compatible with the triangulation.

(ii) For an exact functor $F\colon \mathcal{C} \to \mathcal{D}$ between triangulated categories we have $F(\mathcal{N}) = 0$ if and only if F inverts the morphisms in $S_{\mathcal{N}}$.

(iii) Precomposition with the localization functor

$$Q\colon \mathcal{C} \longrightarrow \mathcal{C}/\mathcal{N} := \mathcal{C}_{S_{\mathcal{N}}}$$

induces a fully faithful embedding $\operatorname{Fun}^{\triangle}(\mathcal{C}/\mathcal{N}, \mathcal{D}) \hookrightarrow \operatorname{Fun}^{\triangle}(\mathcal{C}, \mathcal{D})$ with essential image consisting of those exact functors $F\colon \mathcal{C} \to \mathcal{D}$ satisfying $F(\mathcal{N}) = 0$.

In other words, the following universal property is satisfied: For every exact functor $F\colon \mathcal{C} \to \mathcal{D}$ such that $F(\mathcal{N}) = 0$, there exists a pair $(\overline{F}, \mu)$ consisting of an exact functor $\overline{F}\colon \mathcal{C}/\mathcal{N} \to \mathcal{D}$ and an exact natural isomorphism $\mu\colon F \xrightarrow{\sim} \overline{F} \circ Q$; for any other such pair $(\overline{F}', \mu')$, there exists a unique exact natural isomorphism $\theta\colon \overline{F} \xrightarrow{\sim} \overline{F}'$ such that $\mu' = \theta Q \circ \mu$.

Proof. We first prove (i). Since $0 \in \mathcal{N}$, Proposition 5.3(i) shows that $S_{\mathcal{N}}$ contains all isomorphisms, so that (S1) is satisfied.

Let $f\colon X \to Y$ and $g\colon Y \to Z$ be two morphisms in $S_{\mathcal{N}}$. Embed f, g and $g \circ f$ into distinguished triangles $X \xrightarrow{f} Y \to Z' \to T(X)$, $Y \xrightarrow{g} Z \to X' \to T(Y)$ and $X \xrightarrow{g\circ f} Z \to Y' \to T(X)$, respectively. Then the octahedral axiom provides a distinguished triangle $Z' \to Y' \to X' \to T(X')$. Now $f, g \in S_{\mathcal{N}}$ implies $Z', X' \in \mathcal{N}$. Since $\mathcal{N}$ is triangulated, we deduce $Y' \in \mathcal{N}$, but this means $g \circ f \in S_{\mathcal{N}}$ verifying (S2).

We now verify (S3), so let $Y \xleftarrow{f} X \xrightarrow{s} X'$ with $s \in S_{\mathcal{N}}$. By (T1) and Proposition 4.10 we have a distinguished triangle $X'' \xrightarrow{s''} X \xrightarrow{s} X' \xrightarrow{s'} T(X'')$, where $X'' \in \mathcal{N}$ since $s \in S_{\mathcal{N}}$. Complete fs'' to a distinguished triangle (fs'', t, h) and consider the commutative diagram with solid arrows

$$\begin{tikzcd} X'' \arrow[r, "s''"] \arrow[d, equal] & X \arrow[r, "s"] \arrow[d, "f"] & X' \arrow[r, "s'"] \arrow[d, dashed, "g"] & T(X'') \arrow[d, equal] \\ X'' \arrow[r, "fs''"'] & Y \arrow[r, "t"'] & Z \arrow[r, "h"'] & T(X''). \end{tikzcd}$$

By Proposition 4.4 we find g making the whole diagram commute. Now $X'' \in \mathcal{N}$ implies $t \in S_{\mathcal{N}}$, which shows that (S3) is satisfied.

In order to verify (S4), let $f, g\colon X \rightrightarrows Y$ be two morphisms in $\mathcal{C}$ and $s\colon X' \to X$ in $S_{\mathcal{N}}$ such that $f \circ s = g \circ s$. Put $h \coloneqq f - g$, so that $h \circ s = 0$. We need to find $t \in S_{\mathcal{N}}$ with $t \circ h = 0$. By (T1) we may embed s in a distinguished triangle $X' \xrightarrow{s} X \xrightarrow{s'} X'' \to T(X')$. Since $\mathrm{Hom}_{\mathcal{C}}(-, Y)$ is cohomological by Proposition 4.7, we obtain an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(X'', Y) \xrightarrow{s'^*} \mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{s^*} \mathrm{Hom}_{\mathcal{C}}(X', Y).$$

Since $s^*(h) = h \circ s = 0$, we find $h' \in \mathrm{Hom}_{\mathcal{C}}(X'', Y)$ with $h' \circ s' = s'^*(h') = h$. By (T1) we may complete h' to a distinguished triangle $X'' \xrightarrow{h'} Y \xrightarrow{t} Z \to T(X'')$ with $t \in S_{\mathcal{N}}$ because $X'' \in \mathcal{N}$. By Proposition 4.7(i) we have $t \circ h' = 0$ and hence $t \circ h = t \circ h' \circ s' = 0$, which shows that (S4) is satisfied.

Axiom (S5) is immediate from the fact that $\mathcal{N}$ is a triangulated subcategory.

Finally, let us check axiom (S6). Consider a commutative diagram with solid arrows

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[r, "h"] \arrow[d, dashed, "u"] & T(X) \arrow[d, "T(s)"] \\ X' \arrow[r, "f'"'] & Y' \arrow[r, "g'"'] & Z' \arrow[r, "h'"'] & T(X'), \end{tikzcd}$$

where $s, t \in S_{\mathcal{N}}$. Then by Lemma 4.13 we can complete the diagram to a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "s"'] & Y \arrow[r, "g"] \arrow[d, "t"] & Z \arrow[r, "h"] \arrow[d, "u"] & T(X) \arrow[d, "T(s)"] \\
X' \arrow[r, "f'"] \arrow[d, "s'"'] & Y' \arrow[r, "g'"] \arrow[d, "t'"] & Z' \arrow[r, "h'"] \arrow[d, "u'"] & T(X') \arrow[d, "T(s')"] \\
X'' \arrow[r, "f''"] \arrow[d, "s''"'] & Y'' \arrow[r, "g''"] \arrow[d, "t''"] & Z'' \arrow[r, "h''"] \arrow[d, "u''"] & T(X'') \arrow[d, "T(s'')"] \\
T(X) \arrow[r, "T(f)"'] & T(Y) \arrow[r, "T(g)"'] & T(Z) \arrow[r, "T(h)"'] & T^2(X),
\end{tikzcd}$$

where, in particular, the third row and column are distinguished triangles. Since $s, t \in S_{\mathcal{N}}$, we have $X'', Y'' \in \mathcal{N}$. As $\mathcal{N}$ is triangulated, we deduce $Z'' \in \mathcal{N}$ and hence $u \in S_{\mathcal{N}}$. Hence, axiom (S6) is satisfied. We conclude that $S_{\mathcal{N}}$ is a left multiplicative system compatible with the triangulation. Since $\mathcal{N}^{\mathrm{op}} \subseteq \mathcal{C}^{\mathrm{op}}$ is a triangulated subcategory, we deduce also that $S_{\mathcal{N}}$ is right multiplicative.

We next prove (ii), so let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor between triangulated categories. Then for each distinguished triangle $X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} T(X)$ in $\mathcal{C}$ the triangle

$$F(X) \xrightarrow{F(u)} F(Y) \xrightarrow{F(v)} F(Z) \longrightarrow TF(X)$$

is distinguished. Now we have

$$\begin{array}{rcll} u \in S_{\mathcal{N}} & \iff & Z \in \mathcal{N}, & \text{and} \\ F(u) \text{ is invertible} & \iff & F(Z) = 0 & \text{(by Proposition 5.3),} \end{array}$$

from which the claim is obvious.

Finally, (iii) is a reformulation of Theorem 11.2(iii) using (ii). □

Remark 11.4. Let $(\mathcal{C}, T)$ be a triangulated category.

(i) For any exact functor $F\colon \mathcal{C} \to \mathcal{D}$ we denote by $\operatorname{Ker}(F) \subseteq \mathcal{C}$ the full subcategory spanned by the objects X with $F(X) = 0$. Then $\operatorname{Ker}(F)$ is a thick triangulated subcategory.

(ii) Let $\mathcal{N} \subseteq \mathcal{C}$ be a triangulated subcategory and denote by $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ the associated localization functor. Then $\operatorname{Ker}(Q)$ is the thick closure of $\mathcal{N}$.

Proof. For part (i), note that F is additive by Lemma 6.3. Hence if $X, Y \in \mathcal{C}$ with $X \oplus Y \in \operatorname{Ker}(F)$, then $F(X) \oplus F(Y) = F(X \oplus Y) = 0$. We deduce that $F(X) = F(Y) = 0$, because the composite $F(X) \xrightarrow{F(i_X)} F(X \oplus Y) \xrightarrow{F(p_X)} F(X)$ is both zero and the identity.

Let us prove (ii). We already know that $\operatorname{Ker}(Q)$ is a thick triangulated subcategory containing $\mathcal{N}$. In order to show that $\operatorname{Ker}(Q)$ is contained in the thick closure of $\mathcal{N}$, let $X \in \operatorname{Ker}(Q)$. We need to find $Y \in \mathcal{C}$ such that $X \oplus Y \in \mathcal{N}$. Note that $Q(X \to 0)$ is an isomorphism in $\mathcal{C}/\mathcal{N}$. By Corollary 10.6(iii) we find $Y \in \mathcal{C}$ such that $X \xrightarrow{0} T(Y)$ lies in $S_{\mathcal{N}}$; in particular, $Y \in \operatorname{Ker}(Q)$. But note that we have a distinguished triangle $Y \to X \oplus Y \to X \xrightarrow{0} T(Y)$ by Proposition 5.3(ii). We thus deduce $X \oplus Y \in \mathcal{N}$ as desired. □

Exercise 11.5. Let $(\mathcal{C}, T)$ be a triangulated category and let $\mathcal{N}, \mathcal{D} \subseteq \mathcal{C}$ be triangulated subcategories. Suppose that one of the following conditions is satisfied:

(a) Every map $N \to D$ with $N \in \mathcal{N}$ and $D \in \mathcal{D}$ factors through an object of $\mathcal{N} \cap \mathcal{D}$.

(b) Every map $D \to N$ with $N \in \mathcal{N}$ and $D \in \mathcal{D}$ factors through an object of $\mathcal{N} \cap \mathcal{D}$.

Show that the induced functor $\mathcal{D}/\mathcal{N} \cap \mathcal{D} \hookrightarrow \mathcal{C}/\mathcal{N}$ is fully faithful.

Chapter 4

The Derived Category and Derived Functors

§12. The derived category

Let us come back to the derived category. Recall from Example 8.5 that for an abelian category $\mathcal{A}$ we defined the derived category as

$$\mathsf{D}(\mathcal{A}) \coloneqq \mathsf{C}(\mathcal{A})[\mathrm{qis}^{-1}],$$

where qis denotes the class of quasi-isomorphisms in $\mathsf{C}(\mathcal{A})$ (cf. Definition 3.8). We similarly defined the bounded versions $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ and $\mathsf{D}^b(\mathcal{A})$. We now want to put a triangulated structure on $\mathsf{D}(\mathcal{A})$ using the machinery developed in the previous section.

Lemma 12.1. *Let $\mathcal{A}$ be an abelian category. The localization $Q\colon \mathsf{C}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ factors through $\mathsf{K}(\mathcal{A})$.*

Proof. It suffices to prove the following statement: If $f\colon X \to Y$ is a morphism in $\mathsf{C}(\mathcal{A})$ which is null homotopic, then $Q(f) = 0$. By Exercise 2.14(c), f factors as $X \xrightarrow{\iota_X} \mathrm{Mc}(\mathrm{id}_X) \xrightarrow{(s,f)} Y$, where s is a null homotopy for f. By Proposition 2.13(i), the complex $\mathrm{Mc}(\mathrm{id}_X)$ is acyclic, that is, the map $\mathrm{Mc}(\mathrm{id}_X) \to 0$ is a quasi-isomorphism. We deduce that $Q(f)$ factors through $Q(\mathrm{Mc}(\mathrm{id}_X)) \cong 0$, hence is zero. $\square$

Lemma 12.2. *Let $(\mathcal{C}, T)$ be a triangulated category, $\mathcal{A}$ an abelian category, and $H\colon \mathcal{C} \to \mathcal{A}$ a cohomological functor. Let:*

- *S be the class of morphisms f in $\mathcal{C}$ such that $H(T^i(f))$ is an isomorphism for all $i \in \mathbb{Z}$.*
- *$\mathcal{N} \subseteq \mathcal{C}$ be the full subcategory spanned by those objects N with $H(T^i(N)) = 0$ for all $i \in \mathbb{Z}$.*

Then:

(i) $\mathcal{N}$ is a triangulated subcategory of $\mathcal{C}$.

(ii) $S = S_{\mathcal{N}}$. In other words: If $f\colon X \to Y$ is a morphism in $\mathcal{C}$ and $X \xrightarrow{f} Y \to N \to T(X)$ is a distinguished triangle, then f lies in S if and only if N lies in $\mathcal{N}$.

(iii) S is a left and right multiplicative system which is compatible with the triangulation.

Proof. From the long exact sequence

$$\cdots \longrightarrow H(T^i(X)) \xrightarrow{H(T^i(f))} H(T^i(Y)) \longrightarrow H(T^i(N)) \longrightarrow H(T^{i+1}(X)) \longrightarrow \cdots$$

associated with a distinguished triangle $X \xrightarrow{f} Y \to N \to T(X)$ we make the following observations:

(a) If $X, Y \in \mathcal{N}$, then $N \in \mathcal{N}$.

(b) $f \in S$ if and only if $N \in \mathcal{N}$.

Now, (i) follows from (a), (ii) follows from (b), and (iii) follows from (i), (ii) and Theorem 11.3(i). □

Proposition 12.3. *Let $\mathcal{A}$ be an abelian category.*

(i) qis *is a left and right multiplicative system in* $\mathsf{K}(\mathcal{A})$ *compatible with the triangulation.*

(ii) $\mathsf{K}(\mathcal{A})[\mathrm{qis}^{-1}]$ *is a triangulated category.*

(iii) The induced functor $\mathsf{D}(\mathcal{A}) \xrightarrow{\sim} \mathsf{K}(\mathcal{A})[\mathrm{qis}^{-1}]$ *is an equivalence of categories. In particular,* $\mathsf{D}(\mathcal{A})$ *is a triangulated category, where the distinguished triangles are those which are isomorphic to* $X \xrightarrow{f} Y \to \mathrm{Mc}(f) \to X[1]$ *for some morphism of complexes* $f\colon X \to Y$.

(iv) We have $\mathsf{D}(\mathcal{A}) = \mathsf{K}(\mathcal{A})/\mathsf{K}(\mathcal{A})_{\mathrm{acyc}}$, *where* $\mathsf{K}(\mathcal{A})_{\mathrm{acyc}}$ *denotes the full subcategory of* $\mathsf{K}(\mathcal{A})$ *spanned by the acyclic complexes.*

Similar statements hold for $\mathsf{D}^+(\mathcal{A})$, $\mathsf{D}^-(\mathcal{A})$ *and* $\mathsf{D}^b(\mathcal{A})$.

Proof. Part (i) follows from Lemma 12.2 by observing that $\mathrm{qis} = S_{\mathsf{K}(\mathcal{A})_{\mathrm{acyc}}}$. Hence, (ii) follows from Theorem 11.2. Given (iii), part (iv) is a consequence of Theorem 11.3 using $\mathrm{qis} = S_{\mathsf{K}(\mathcal{A})_{\mathrm{acyc}}}$ by Lemma 12.2. It remains to prove (iii). Note first that Lemma 12.1 provides a factorization $\mathsf{C}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. Given any category $\mathcal{E}$, we get a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}(\mathsf{D}(\mathcal{A}), \mathcal{E}) \arrow[r] \arrow[dr, hook] & \mathrm{Fun}(\mathsf{K}(\mathcal{A}), \mathcal{E}) \arrow[d, hook] \\
& \mathrm{Fun}(\mathsf{C}(\mathcal{A}), \mathcal{E}).
\end{tikzcd}$$

It is easy to see that the right vertical functor is fully faithful. Hence the horizontal functor is fully faithful, and the essential image consists of those functors which invert quasi-isomorphisms. Hence the functor $\mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ exhibits $\mathsf{D}(\mathcal{A})$ as the localization of $\mathsf{K}(\mathcal{A})$ with respect to the quasi-isomorphisms. □

Example 12.4. Let $\mathcal{A}$ be an abelian category and let $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ a short exact sequence of complexes. Then there exists a morphism $d\colon C \to A[1]$ in $\mathsf{D}(\mathcal{A})$ such that

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{d} A[1]$$

is a distinguished triangle in $\mathsf{D}(\mathcal{A})$. Observe that g factors as a composition $B \xrightarrow{(0,\mathrm{id})} \mathrm{Mc}(f) \xrightarrow{(0,g)} C$ as $g \circ f = 0$. Now, from the long exact sequence in cohomology and the five lemma it follows that $(0, g)$ is a quasi-isomorphism. This implies the claim.

Remark 12.5. For morphisms $f, g\colon X \rightrightarrows Y$ of complexes we have the following *strict* implications:

$$\begin{aligned} f = g \text{ in } \mathsf{C}(\mathcal{A}) &\implies f = g \text{ in } \mathsf{K}(\mathcal{A}), \\ &\implies f = g \text{ in } \mathsf{D}(\mathcal{A}), \\ &\implies \mathrm{H}^i(f) = \mathrm{H}^i(g) \text{ for all } i \in \mathbb{Z}. \end{aligned}$$

To see the strictness, we consider the following counter-examples:

(1) Let $X \in \mathsf{C}(\mathcal{A})$ be a non-zero complex. The identity on $\mathrm{Mc}(\mathrm{id}_X)$ is not zero in $\mathsf{C}(\mathcal{A})$ but becomes zero in $\mathsf{K}(\mathcal{A})$ by Proposition 2.13(i).

(2) Consider the complex $X = [0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to \mathbb{Z}/2 \to 0]$ in $\mathsf{K}(\mathcal{A})$. Since X is acyclic, the identity on X is zero in $\mathsf{D}(\mathcal{A})$; but it is non-zero in $\mathsf{K}(\mathcal{A})$: If X were contractible, there would exist the dashed maps in the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, equal] & \mathbb{Z} \arrow[r, "\pi"] \arrow[d, equal] \arrow[dl, dashed, "s"'] & \mathbb{Z}/2 \arrow[r] \arrow[d, equal] \arrow[dl, dashed, "t"'] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"'] & \mathbb{Z} \arrow[r] & \mathbb{Z}/2 \arrow[r, "\pi"'] & 0 \end{tikzcd}$$

such that $2s(1) = s(2) = 1$, $2s + t\pi = \mathrm{id}_{\mathbb{Z}}$ and $\pi t = \mathrm{id}_{\mathbb{Z}/2}$; but this is absurd.

(3) Consider the complexes $[0 \to \mathbb{Z} \xrightarrow{2} \mathbb{Z} \to 0]$ and $\mathbb{Z}[1]$ in $\mathsf{C}(\mathsf{Ab})$. The map f of complexes

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z} \arrow[r, "2"] \arrow[d, equal] & \mathbb{Z} \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathbb{Z} \arrow[r] & 0 \arrow[r] & 0 \end{tikzcd}$$

is clearly zero on cohomology, but it is non-zero in $\mathsf{D}(\mathsf{Ab})$: otherwise, by Corollary 10.6(ii), there would exist a quasi-isomorphism $s\colon \mathbb{Z}[1] \to Y$ of complexes such that the composite $s \circ f$ is zero in $\mathsf{C}(\mathsf{Ab})$; but this contradicts the fact that $s^{-1} \circ f^{-1} = s^{-1} \neq 0$ (because $\mathrm{H}^{-1}(s) \neq 0$).

This example generalizes: We will see later (in Theorem 16.7) that for all $X, Y \in \mathcal{A}$ and $n \in \mathbb{Z}$ we have $\mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n]) = \mathrm{Ext}^n_{\mathcal{A}}(X, Y)$.

Definition 12.6. Let $\mathcal{A}$ be an abelian category and $X \in \mathsf{C}(\mathcal{A})$ be a complex. We define the *truncated complexes*

$$\begin{aligned} \tau^{\leq n} X &:= [\cdots \to X^{n-2} \xrightarrow{d^{n-2}} X^{n-1} \to \mathrm{Ker}(d^n) \to 0 \to \cdots], \\ \tau^{\geq n} X &:= [\cdots \to 0 \to \mathrm{Coker}(d^{n-1}) \to X^{n+1} \xrightarrow{d^{n+1}} X^{n+2} \to \cdots], \\ \tilde{\tau}^{\leq n} X &:= [\cdots \to X^{n-1} \to X^n \to \mathrm{Im}(d^n) \to 0 \to \cdots], \\ \tilde{\tau}^{\geq n} X &:= [\cdots \to 0 \to \mathrm{Im}(d^{n-1}) \to X^n \to X^{n+1} \to \cdots], \\ \sigma^{\geq n} X &:= [\cdots \to 0 \to X^n \to X^{n+1} \to \cdots], \\ \sigma^{\leq n} X &:= [\cdots \to X^{n-1} \to X^n \to 0 \to \cdots]. \end{aligned}$$

The following obvious properties are satisfied:

(a) $\tau^{\leq n}\tau^{\geq n}X \cong \mathrm{H}^n(X)[-n]$ for all $n \in \mathbb{Z}$;

(b) We have canonical morphisms $\sigma^{\geq n-1}X \to X \to \tilde{\tau}^{\geq n}X \to \tau^{\geq n}X$; the first two maps are quasi-isomorphisms in degrees $\geq n$ and the third is a quasi-isomorphism.

Similarly, we have canonical morphisms $\tau^{\leq n}X \to \tilde{\tau}^{\leq n}X \to X \to \sigma^{\leq n+1}X$, where the first map is a quasi-isomorphism and the last two maps are quasi-isomorphisms in degrees $\leq n$.

(c) We have canonical short exact sequences

$$0 \to \tau^{\leq n}X \to X \to \tilde{\tau}^{\geq n+1}X \to 0,$$
$$0 \to \tilde{\tau}^{\leq n}X \to X \to \tau^{\geq n+1}X \to 0,$$
$$0 \to \sigma^{\geq n}X \to X \to \sigma^{\leq n-1}X \to 0.$$

By Example 12.4 these induce distinguished triangles in $\mathsf{D}(\mathcal{A})$:

$$\tau^{\leq n}X \to X \to \tau^{\geq n+1}X \to \tau^{\leq n}X[1],$$
$$\sigma^{\geq n}X \to X \to \sigma^{\leq n-1}X \to \sigma^{\geq n}X[1].$$

Proposition 12.7. *Let $\mathcal{A}$ be an abelian category. The functor*

$$\mathcal{A} \longrightarrow \mathsf{D}(\mathcal{A}), \qquad A \longmapsto A[0],$$

is fully faithful. The essential image consists of those complexes X such that $\mathrm{H}^i(X) = 0$ for all $i \neq 0$.

Proof. Let $A, B \in \mathcal{A}$. Injectivity of the map $\mathrm{Hom}_{\mathcal{A}}(A,B) \to \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(A[0], B[0])$ follows from Corollary 10.6(ii): If $f\colon A[0] \to B[0]$ is the zero map, then there exists a quasi-isomorphism $s\colon B[0] \to X^\bullet$ such that $s \circ f = 0$ in $\mathsf{C}(\mathcal{A})$. But then $\mathrm{H}^0(s \circ f) = \mathrm{H}^0(s) \circ f\colon A \to \mathrm{H}^0(X)$ is zero. As $\mathrm{H}^0(s)$ is an isomorphism, we deduce $f = 0$.

To prove surjectivity, consider a morphism $A[0] \to B[0]$ in $\mathsf{D}(\mathcal{A})$ represented by a roof $A[0] \xrightarrow{f} X \xleftarrow{s} B[0]$, where s is a quasi-isomorphism of complexes. We have a commutative diagram

$$\begin{tikzcd}
 & X \arrow[d] & \\
A \arrow[ur, "f"] \arrow[dr, "\mathrm{H}^0(f)"'] & \tau^{\geq 0}X & B, \arrow[ul, "s"'] \arrow[dl, "\cong"] \\
 & \mathrm{H}^0(X), \arrow[u] &
\end{tikzcd}$$

where the maps $X \to \tau^{\geq 0}X \leftarrow \mathrm{H}^0(X)$ are quasi-isomorphisms. This shows that every map $A[0] \to B[0]$ is represented by a morphism in $\mathcal{A}$. $\square$

Lemma 12.8. *Let $\mathcal{A}$ be an abelian category. For $* \in \{+,-,b\}$ the canonical functor $\mathsf{D}^*(\mathcal{A}) \hookrightarrow \mathsf{D}(\mathcal{A})$ is fully faithful and we have $\mathsf{D}^+(\mathcal{A}) \cap \mathsf{D}^-(\mathcal{A}) = \mathsf{D}^b(\mathcal{A})$ inside $\mathsf{D}(\mathcal{A})$.*

Moreover, the essential image of $\mathsf{D}^+(\mathcal{A})$ (resp. $\mathsf{D}^-(\mathcal{A})$) in $\mathsf{D}(\mathcal{A})$ consists of those objects X with $\mathrm{H}^i(X) = 0$ for $i \ll 0$ (resp. $i \gg 0$).

Proof. Observe that the obvious functor $\mathsf{K}^*(\mathcal{A}) \hookrightarrow \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ inverts quasi-isomorphisms and hence induces a well-defined functor $\mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$. In order to check fully faithfulness, we use the criterion in Proposition 10.8 (and its dual).

If $* = +$, it remains to show that for every quasi-isomorphism $s\colon X \to X'$ in $\mathsf{K}(\mathcal{A})$ with $X \in \mathsf{K}^+(\mathcal{A})$, there exists a map $g\colon X' \to Y$ of complexes with $Y \in \mathsf{K}^+(\mathcal{A})$ such that $g \circ s$ is a quasi-isomorphism. We fix $n \in \mathbb{Z}$ such that $\mathrm{H}^i(X) = 0$ for all $i < n$. Since s is a quasi-isomorphism, we also have $\mathrm{H}^i(X') = 0$ for all $i < n$. The obvious map $g\colon X' \to \tau^{\geq n} X'$ is a quasi-isomorphism, hence so is $g \circ s$. We deduce that $\mathsf{D}^+(\mathcal{A}) \hookrightarrow \mathsf{D}(\mathcal{A})$ is fully faithful and the essential image consists of those complexes X with $\mathrm{H}^i(X) = 0$ for all $i \ll 0$. The same argument shows that $\mathsf{D}^b(\mathcal{A}) \hookrightarrow \mathsf{D}^-(\mathcal{A})$ is fully faithful.

If $* = -$, then we need to show that for every quasi-isomorphism $s\colon Y' \to Y$ with $Y \in \mathsf{K}^-(\mathcal{A})$ there exists $g\colon X \to Y'$ with $X \in \mathsf{K}^-(\mathcal{A})$ such that $s \circ g$ is a quasi-isomorphism. Fix $n \in \mathbb{Z}$ such that $\mathrm{H}^i(Y) = 0$ for all $i > n$. Then the obvious map $g\colon \tau^{\leq n} Y' \to Y'$ is a quasi-isomorphism, hence so is $s \circ g$. Therefore, the functor $\mathsf{D}^-(\mathcal{A}) \hookrightarrow \mathsf{D}(\mathcal{A})$ is fully faithful for $* \in \{-, +\}$. The remaining assertions are clear. □

Lemma 12.9. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor between abelian categories. The following statements are equivalent:*

(a) F is exact.

(b) If $X \in \mathsf{C}(\mathcal{A})$ is an acyclic complex, then $F(X)$ is acyclic.

(c) If $f\colon X \to Y$ is a quasi-isomorphism in $\mathsf{K}(\mathcal{A})$, then $\mathsf{K}(F)(f)$ is a quasi-isomorphism in $\mathsf{K}(\mathcal{B})$.

If these conditions are satisfied, then F induces an exact functor $\mathsf{D}(F)\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$.

Proof. The equivalence of (a) and (b) is immediate from the definition. We now prove the equivalence of (b) and (c). Let $f\colon X \to Y$ be a morphism of complexes. Note that $\mathsf{K}(F)$ is exact by Example 6.4. The claim now follows from the observation that f is a quasi-isomorphism if and only if $\mathrm{Mc}(f)$ is acyclic Lemma 12.2(iv). The final claim follows from (c) and the universal property of $\mathsf{D}(\mathcal{A})$. □

We will next discuss a generalization of Lemma 12.9 to general additive functors $F\colon \mathcal{A} \to \mathcal{B}$. Concretely, we have a diagram with solid arrows

$$\begin{tikzcd} \mathsf{K}(\mathcal{A}) \arrow[r, "\mathsf{K}(F)"] \arrow[d, "Q_{\mathcal{A}}"'] & \mathsf{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathsf{D}(\mathcal{A}) \arrow[r, dashed] & \mathsf{D}(\mathcal{B}). \end{tikzcd}$$

It is generally not true that $Q_{\mathcal{B}} \circ \mathsf{K}(F)$ inverts quasi-isomorphisms, in which case there is no dashed arrow making the whole diagram commute. However, in good situations there exists a universal functor $\mathrm{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$, called the *right derived functor* of F, which "approximately" makes the diagram commutative. We will study this phenomenon in greater generality in the next section.

§13. Kan extensions

Definition 13.1. Let $\varphi\colon \mathcal{C} \to \mathcal{D}$ be a functor and $\mathcal{E}$ a category. Consider the induced functor

$$\begin{aligned} \varphi^*\colon \operatorname{Fun}(\mathcal{D},\mathcal{E}) &\longrightarrow \operatorname{Fun}(\mathcal{C},\mathcal{E}),\\ G &\longmapsto \varphi^*(G) \coloneqq G \circ \varphi. \end{aligned}$$

Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor.

(a) We say that F admits a *left Kan extension along* φ if the following equivalent definitions are satisfied:

- φ^* admits a (partial) left adjoint at F.
- there exists a pair $(\overline{F}, \eta)$ consisting of a functor $\overline{F}\colon \mathcal{D} \to \mathcal{E}$ and a natural transformation $\eta\colon F \to \overline{F} \circ \varphi$ such that the composite

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D},\mathcal{E})}(\overline{F}, G) \xrightarrow{\varphi} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(\overline{F}\circ\varphi, G\circ\varphi) \xrightarrow{\eta^*} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(F, G\circ\varphi)$$

is bijective for all $G \in \operatorname{Fun}(\mathcal{D},\mathcal{E})$.
In other words, for all functors $G\colon \mathcal{D} \to \mathcal{E}$ and all natural transformations $\nu\colon F \to G \circ \varphi$, there exists a unique natural transformation $\alpha\colon \overline{F} \to G$ such that $\nu = \alpha\varphi \circ \eta$:

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "\varphi"'] & \arrow[d, phantom, "\Downarrow\eta"] & \mathcal{E} \\ & \mathcal{D} \arrow[ur, dashed, "\overline{F}"] \arrow[ur, bend right=60, "G"'] \arrow[ur, phantom, bend right=30, "\Downarrow\exists!\alpha"] & \end{tikzcd}$$

In this case, we write $\varphi_! F \coloneqq \overline{F}$ and call it the left Kan extension of F along φ.

(b) We say that a functor $H\colon \mathcal{E} \to \mathcal{F}$ *preserves* the left Kan extension of $F\colon \mathcal{C} \to \mathcal{E}$ along φ if the natural transformation $H\eta\colon H \circ F \to H \circ \varphi_! F \circ \varphi$ exhibits $H \circ \varphi_! F$ as the left Kan extension of $H \circ F$ along φ; in other words, the induced natural transformation $\varphi_!(H \circ F) \xrightarrow{\sim} H \circ \varphi_! F$ is an isomorphism.

We similarly say that F admits a *right Kan extension* along φ if there exists a functor $\overline{F}\colon \mathcal{D} \to \mathcal{E}$ and a natural transformation $\varepsilon\colon \overline{F} \circ \varphi \to F$ such that the composite

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D},\mathcal{E})}(G, \overline{F}) \xrightarrow{\varphi} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(G\circ\varphi, \overline{F}\circ\varphi) \xrightarrow{\varepsilon_*} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(G\circ\varphi, F)$$

is bijective for all $G \in \operatorname{Fun}(\mathcal{D},\mathcal{E})$. In this case, we write $\varphi_* F \coloneqq \overline{F}$. A functor $H\colon \mathcal{E} \to \mathcal{F}$ preserves the right Kan extension of F along φ if $H\varepsilon\colon H \circ \varphi_* F \circ \varphi \to H \circ F$ exhibits $H \circ \varphi_* F$ as the right Kan extension of $H \circ F$ along φ.

Remark 13.2. It follows from the definition of Kan extensions that, if $\alpha\colon F_1 \to F_2$ is a natural transformation of functors $\mathcal{C} \to \mathcal{E}$ which admit a left Kan extension along $\varphi\colon \mathcal{C} \to \mathcal{D}$, then there is a unique natural map $\varphi_!\alpha\colon \varphi_! F_1 \to \varphi_! F_2$ making the diagram

$$\begin{tikzcd} F_1 \arrow[r, "\alpha"] \arrow[d, "\eta_1"'] & F_2 \arrow[d, "\eta_2"] \\ \varphi_! F_1 \circ \varphi \arrow[r, "\varphi_!\alpha\circ\varphi"'] & \varphi_! F_2 \circ \varphi \end{tikzcd}$$

commute. It follows easily that $\varphi_!$ is functorial. Similarly, φ_* is functorial.

Example 13.3. A left Kan extension of a functor $F\colon \mathcal{C} \to \mathcal{D}$ along $\mathcal{C} \to *$ is a pair $(\operatorname{colim} F, \eta)$ consisting of a colimit of F together with the universal cocone $\eta\colon F \to \mathrm{const}_{\operatorname{colim} F}$. Similarly, a right Kan extension along $\mathcal{C} \to *$ is a limit.

Exercise 13.4. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and $c \in \mathcal{C}$. Compute the left Kan extension of $\mathrm{Hom}_{\mathcal{C}}(c, -)\colon \mathcal{C} \to \mathsf{Set}$ along F.

Exercise 13.5. Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor. Show that the assignment $\varphi \mapsto \varphi_! F$ is contravariantly functorial in those φ along which F admits a left Kan extension.

We have the following generalization of the fact that left adjoints preserve colimits:

Lemma 13.6. *Left adjoints preserve left Kan extensions. Right adjoints preserve right Kan extensions.*

Proof. We only prove the statement for left adjoints. Consider a diagram

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "\varphi"'] & \mathcal{E} \arrow[r, "L"] & \mathcal{F} \\ \mathcal{D} \arrow[ur, dashed, "\varphi_! F"'] & & \end{tikzcd}$$

of functors, where L admits a right adjoint $R\colon \mathcal{F} \to \mathcal{E}$. For every functor $G\colon \mathcal{D} \to \mathcal{F}$ we compute

$$\mathrm{Hom}(L\varphi_! F, G) \cong \mathrm{Hom}(\varphi_! F, RG) \cong \mathrm{Hom}(F, RG\varphi) \cong \mathrm{Hom}(LF, G\varphi).$$

□

Definition 13.7. Let $\varphi\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{D} \to \mathcal{E}$ be functors and suppose that the left Kan extension $\varphi_! F\colon \mathcal{D} \to \mathcal{E}$ of F along φ exists.

(a) We say that $\varphi_! F$ is *absolute* if it is preserved by *every* functor $H\colon \mathcal{E} \to \mathcal{F}$.

(b) We say that $\varphi_! F$ is *pointwise* if it is preserved by the functors $\mathrm{Hom}_{\mathcal{E}}(-, e)\colon \mathcal{E} \to \mathsf{Set}^{\mathrm{op}}$ for all $e \in \mathcal{E}$.

Similarly, a right Kan extension is called *absolute* if it is preserved by every functor, and is called *pointwise* if it is preserved by the functors $\mathrm{Hom}_{\mathcal{E}}(e, -)\colon \mathcal{E} \to \mathsf{Set}$ for all $e \in \mathcal{E}$.

We have already seen in Section §8 a special type of absolute Kan extension along a localization functor. More precisely, we have:

Lemma 13.8. *Let $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ be a localization functor and $F\colon \mathcal{C} \to \mathcal{D}$ be any functor. The following statements are equivalent:*

(a) The functor F sends morphisms in S to isomorphisms in $\mathcal{D}$.

(b) There exists a functor $\overline{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ and a natural isomorphism $\eta\colon F \xrightarrow{\sim} \overline{F}Q$, and any such pair $(\overline{F}, \eta)$ is an absolute left Kan extension of F along Q.

(c) There exists a functor $\overline{F}\colon \mathcal{C}[S^{-1}] \to \mathcal{D}$ and a natural isomorphism $\varepsilon\colon \overline{F}Q \xrightarrow{\sim} F$, and any such pair $(\overline{F}, \varepsilon)$ is an absolute right Kan extension of F along Q.

Proof. We only prove the equivalence of (a) and (b). The equivalence of (a) and (c) is similar. The implication "(b) $\implies$ (a)" is obvious. Conversely, suppose that F inverts S. Since $Q^*\colon \operatorname{Fun}(\mathcal{C}[S^{-1}], \mathcal{D}) \hookrightarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D})$ is fully faithful and F lies in the essential image, there exists a natural isomorphism $\eta\colon F \xrightarrow{\sim} \overline{F}Q$. Now, for all functors $H\colon \mathcal{D} \to \mathcal{E}$ and $G\colon \mathcal{C}[S^{-1}] \to \mathcal{E}$, the composite

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}[S^{-1}],\mathcal{E})}(H\overline{F}, G) \xrightarrow[\sim]{Q^*} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(H\overline{F}Q, GQ) \xrightarrow[\sim]{\eta^*} \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(HF, GQ)$$

is an isomorphism, which shows that $H\eta$ exhibits $H\overline{F}$ as a left Kan extension of HF along Q. Hence, $\overline{F}$ is absolute. □

Proposition 13.9. *Let $Q\colon \mathcal{C} \to \mathcal{C}[S^{-1}]$ be a localization functor. An object $Y \in \mathcal{C}$ is called* S-local *if for all morphisms $s\colon X \to X'$ in S the induced map $\operatorname{Hom}_{\mathcal{C}}(X', Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}}(X, Y)$ is bijective. Suppose that the following condition is satisfied:*

- *($*$) For every $X \in \mathcal{C}$ there exists a map $f\colon X \to Y$ such that Y is S-local and $Q(f)$ is invertible in $\mathcal{C}[S^{-1}]$.*

Then:

- *(i) If $Y \in \mathcal{C}$ is S-local, then for all $X \in \mathcal{C}$ the map*

$$\operatorname{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(Q(X), Q(Y))$$

induced by Q is bijective.

- *(ii) Q is a Bousfield localization. More precisely, Q admits a fully faithful right adjoint $\mathcal{C}[S^{-1}] \hookrightarrow \mathcal{C}$ whose essential image is spanned by the S-local objects.*

Proof. We first prove (i). Note that also $Q^{\mathrm{op}}\colon \mathcal{C}^{\mathrm{op}} \to \mathcal{C}[S^{-1}]^{\mathrm{op}}$ is a localization functor. For every functor $G\colon \mathcal{C}[S^{-1}]^{\mathrm{op}} \to \mathsf{Set}$ the Yoneda lemma provides isomorphisms

$$\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}^{\mathrm{op}},\mathsf{Set})}\big(\operatorname{Hom}_{\mathcal{C}}(-, Y), GQ\big) = GQ(Y) = \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}[S^{-1}]^{\mathrm{op}},\mathsf{Set})}\big(\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(-, Q(Y)), G\big),$$

which shows that $\eta\colon \operatorname{Hom}_{\mathcal{C}}(-, Y) \to \operatorname{Hom}_{\mathcal{C}[S^{-1}]}(Q(-), Q(Y))$ exhibits $\operatorname{Hom}_{\mathcal{C}[S^{-1}]}(-, Q(Y))$ as the left Kan extension of $\operatorname{Hom}_{\mathcal{C}}(-, Y)$ along Q^{op}. But $\operatorname{Hom}_{\mathcal{C}}(-, Y)$ inverts the morphisms in S and hence Lemma 13.8 shows that η is an isomorphism.

For part (ii), let $\mathcal{C}^{\mathrm{loc}} \subseteq \mathcal{C}$ be the full subcategory spanned by the S-local objects. Then (i) shows that the restriction

$$Q\big|_{\mathcal{C}^{\mathrm{loc}}}\colon \mathcal{C}^{\mathrm{loc}} \longhookrightarrow \mathcal{C}[S^{-1}]$$

is fully faithful. By ($*$), it is also essentially surjective, hence an equivalence of categories. Now the composite $\mathcal{C}[S^{-1}] \xrightarrow{\sim} \mathcal{C}^{\mathrm{loc}} \subseteq \mathcal{C}$ defines a fully faithful right adjoint of Q. □

The following result gives a powerful criterion for the existence of absolute Kan extensions:

Proposition 13.10. *Let $\varphi\colon \mathcal{C} \rightleftarrows \mathcal{D} :\psi$ be an adjunction and denote by $\eta\colon \mathrm{id}_{\mathcal{C}} \to \psi\varphi$ and $\varepsilon\colon \varphi\psi \to \mathrm{id}_{\mathcal{D}}$ the unit and counit, respectively.*

(i) For every functor $F\colon \mathcal{C} \to \mathcal{E}$, the natural transformation $F\eta\colon F \to F\psi \circ \varphi$ exhibits $F\psi$ as the left Kan extension of F along φ. Moreover, $\varphi_! F = F\psi$ is absolute and in particular pointwise.

(ii) For every functor $G\colon \mathcal{D} \to \mathcal{E}$, the natural transformation $G\varepsilon\colon G\varphi \circ \psi \to G$ exhibits $G\varphi$ as the right Kan extension of G along ψ. Moreover, $\psi_* G = G\varphi$ is absolute and in particular pointwise.

Proof. We only prove (i), because the other statement is analogous. The adjunction $\varphi \dashv \psi$ induces an adjunction

$$\psi^*\colon \operatorname{Fun}(\mathcal{C}, \mathcal{E}) \rightleftarrows \operatorname{Fun}(\mathcal{D}, \mathcal{E}) : \varphi^*.$$

Indeed, the natural transformations $\eta^*_F\colon F \xrightarrow{F\eta} F\psi\varphi = \varphi^*\psi^*(F)$ (where $F \in \operatorname{Fun}(\mathcal{C}, \mathcal{E})$) and $\varepsilon^*_G\colon \psi^*\varphi^*(G) = G\varphi\psi \xrightarrow{G\varepsilon} G$ (where $G \in \operatorname{Fun}(\mathcal{D}, \mathcal{E})$) clearly satisfy the triangle identities.

For every functor $H\colon \mathcal{E} \to \mathcal{F}$ the diagram

$$\begin{tikzcd}
\operatorname{Fun}(\mathcal{C}, \mathcal{E}) \arrow[r, shift left, "\psi^*"] \arrow[d, "H_*"'] & \operatorname{Fun}(\mathcal{D}, \mathcal{E}) \arrow[l, shift left, "\varphi^*"] \arrow[d, "H_*"] \\
\operatorname{Fun}(\mathcal{C}, \mathcal{F}) \arrow[r, shift left, "\psi^*"] & \operatorname{Fun}(\mathcal{D}, \mathcal{F}) \arrow[l, shift left, "\varphi^*"]
\end{tikzcd}$$

is commutative. This implies that the Kan extensions are absolute. □

Corollary 13.11. *Let $\varphi\colon \mathcal{C} \xrightarrow{\sim} \mathcal{D}$ be an equivalence of categories. Then every functor $F\colon \mathcal{C} \to \mathcal{E}$ admits an (absolute) left (resp. right) Kan extension along φ given by $\varphi_! F = F\varphi^{-1}$ (resp. $\varphi_* F = F\varphi^{-1}$).*

Proof. Note that φ^{-1} is left and right adjoint to φ (by Observation 7.1 and its dual). Hence the claim follows from Proposition 13.10. □

Exercise 13.12. Show that a functor $F\colon \mathcal{C} \to \mathcal{D}$ admits a right adjoint if and only if the left Kan extension $F_!(\mathrm{id}_{\mathcal{C}})\colon \mathcal{D} \to \mathcal{C}$ exists and is absolute.

It turns out that the only Kan extensions of interest are the pointwise ones. The next result shows that there is an explicit formula to compute pointwise Kan extensions and that every Kan extension is pointwise if $\mathcal{E}$ admits enough (co)limits.

As a preparation we recall the following simple lemma:

Lemma 13.13. *Let $\mathcal{C}$ be a category and $F \in \mathcal{P}(\mathcal{C}) \coloneqq \operatorname{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ a presheaf. We denote by*

$$\mathcal{C}_{/F} \coloneqq \mathcal{C} \times_{\mathcal{P}(\mathcal{C})} \mathcal{P}(\mathcal{C})_{/F}$$

the category of pairs (c, x), where $c \in \mathcal{C}$ and $x \in F(c)$ (equivalently, x corresponds to a map $\tau_{c,x}\colon \operatorname{Hom}(-, c) \to F$ by the Yoneda lemma).

Then the canonical map

$$\operatorname*{colim}_{(c,f) \in \mathcal{C}_{/F}} \operatorname{Hom}_{\mathcal{C}}(-, c) \xrightarrow{\sim} F$$

is an isomorphism.

Similarly, for every $F \in \mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ the canonical map

$$\operatorname*{colim}_{(c,f)\in(\mathcal{C}_{/F})^{\mathrm{op}}} \mathrm{Hom}_{\mathcal{C}}(c, -) \xrightarrow{\sim} F$$

is an isomorphism.

Proof. The second claim follows from the first by replacing $\mathcal{C}$ by $\mathcal{C}^{\mathrm{op}}$ and observing that $\mathcal{C}^{\mathrm{op}}_{/F} = (\mathcal{C}_F)^{\mathrm{op}}$.

A morphism $(c, x) \to (c', x')$ in $\mathcal{C}_{/F}$ is a map $h\colon c \to c'$ making the diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{C}}(-, c) \arrow[d, "h_*"'] \arrow[dr, "\tau_{c,x}"] & \\
 & F \\
\mathrm{Hom}_{\mathcal{C}}(-, c') \arrow[ur, "\tau_{c',x'}"'] &
\end{tikzcd}$$

commute. Hence the map in question is well-defined. It remains to see that for each $A \in \mathcal{C}$ the map

$$\operatorname*{colim}_{(c,x)\in\mathcal{C}_{/F}} \mathrm{Hom}_{\mathcal{C}}(A, c) \to F(A), \qquad g \mapsto \tau_{c,x}(g) = F(g)(x).$$

is bijective.

Observe that $\tau_{A,x}(\mathrm{id}_A) = x$ for every $x \in F(A)$. Hence surjectivity is clear. For injectivity, let $g_i \in \mathrm{Hom}_{\mathcal{C}}(A, c_i)$ for $i = 1, 2$ (viewed as an element of the colimit at $(c_i, x_i) \in \mathcal{C}_{/F}$), such that the images of g_1 and g_2 in $F(A)$ agree, *i.e.*, $\tau_{c_1,x_1}(g_1) = \tau_{c_2,x_2}(g_2) =: y$. We have morphisms $g_i\colon (A, y) \to (c_i, x_i)$ in $\mathcal{C}_{/F}$, and hence $[g_1, (c_1, x_1)] = [\mathrm{id}_A, (A, y)] = [g_2, (c_2, x_2)]$ in the colimit as desired. □

Proposition 13.14. *Let $\varphi\colon \mathcal{C} \to \mathcal{D}$ and $F\colon \mathcal{C} \to \mathcal{E}$ be functors.*

(i) The left Kan extension $\varphi_! F\colon \mathcal{D} \to \mathcal{E}$ of F along φ exists and is pointwise if and only if for all $d \in \mathcal{D}$ the colimit of $\varphi/d \xrightarrow{s} \mathcal{C} \xrightarrow{F} \mathcal{E}$ exists. In this case, the natural map

$$\operatorname{colim}\bigl(\varphi/d \xrightarrow{s} \mathcal{C} \xrightarrow{F} \mathcal{E}\bigr) \xrightarrow{\sim} \varphi_! F(d)$$

is an isomorphism for all $d \in \mathcal{D}$.

(ii) The right Kan extension $\varphi_* F\colon \mathcal{D} \to \mathcal{E}$ of F along φ exists and is pointwise if and only if for all $d \in \mathcal{D}$ the limit of $d/\varphi \xrightarrow{t} \mathcal{C} \xrightarrow{F} \mathcal{E}$ exists. In this case, the natural map

$$\varphi_* F(d) \xrightarrow{\sim} \lim\bigl(d/\varphi \xrightarrow{t} \mathcal{C} \xrightarrow{F} \mathcal{E}\bigr)$$

is an isomorphism.

Proof. We only prove (ii). Statement (i) then follows by passing to opposite categories everywhere (and using $\mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathcal{D}^{\mathrm{op}}) = \mathrm{Fun}(\mathcal{C}, \mathcal{D})^{\mathrm{op}}$).

Fix $d \in \mathcal{D}$. The category $d/\varphi := \mathcal{D}_{d/} \times_{\mathcal{D}} \mathcal{C}$ is the category with objects the pairs (c, f), where $c \in \mathcal{C}$ and $f\colon d \to \varphi(c)$ is a morphism in $\mathcal{D}$; a morphism $(c, f) \to (c', f')$ consists of a map $g\colon c \to c'$ in $\mathcal{C}$ such that $f' = \varphi(g) \circ f$. The functor $t\colon d/\varphi \to \mathcal{C}$ is given by $t(c, f) := c$.

Observe that $d/\varphi = \mathcal{C}_{/\operatorname{Hom}(d,\varphi(-))}$ canonically. Lemma 13.13 shows that the natural transformation

$$\operatorname*{colim}_{(c,f)\in(d/\varphi)^{\mathrm{op}}} \operatorname{Hom}_{\mathcal{C}}(c,-) \xrightarrow{\sim} \operatorname{Hom}_{\mathcal{D}}(d,\varphi(-)) \tag{13.1}$$

of functors $\mathcal{C}\to\mathsf{Set}$ is an isomorphism.

Suppose first that the right Kan extension $(\varphi_*F,(\varphi_*F)\varphi \xrightarrow{\varepsilon} F)$ exists. For every $e\in\mathcal{E}$ we compute

$$\begin{aligned}
\operatorname{Hom}_{\mathcal{E}}(e,\varphi_*F(d)) &\cong \operatorname{Hom}_{\mathrm{Fun}(\mathcal{D},\mathsf{Set})}\big(\operatorname{Hom}_{\mathcal{D}}(d,-),\operatorname{Hom}_{\mathcal{E}}(e,\varphi_*F(-))\big) && \text{(Yoneda)}\\
&\to \operatorname{Hom}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}\big(\operatorname{Hom}_{\mathcal{D}}(d,\varphi(-)),\operatorname{Hom}_{\mathcal{E}}(e,F(-))\big)\\
&\cong \operatorname{Hom}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}\Big(\operatorname*{colim}_{(c,f)\in(d/\varphi)^{\mathrm{op}}}\operatorname{Hom}_{\mathcal{C}}(c,-),\operatorname{Hom}_{\mathcal{E}}(e,F(-))\Big) && \text{(by (13.1))}\\
&\cong \lim_{(c,f)\in d/\varphi}\operatorname{Hom}_{\mathrm{Fun}(\mathcal{C},\mathsf{Set})}\big(\operatorname{Hom}_{\mathcal{C}}(c,-),\operatorname{Hom}_{\mathcal{E}}(e,F(-))\big)\\
&\cong \lim_{(c,f)\in d/\varphi}\operatorname{Hom}_{\mathcal{E}}(e,F(c)) && \text{(Yoneda)}\\
&\cong \operatorname{Hom}_{\mathcal{E}}\Big(e,\lim_{(c,f)\in d/\varphi}F(c)\Big).
\end{aligned}$$

Again by the Yoneda lemma, we obtain a map $\varphi_*F(d)\to\lim_{(c,f)\in d/\varphi}F(c)$, and the construction shows that it is an isomorphism if and only if φ_*F is pointwise.

Conversely, suppose that the limit $\overline{F}(d) := \varprojlim(d/\varphi \xrightarrow{t} \mathcal{C} \xrightarrow{F} \mathcal{E})$ exists for all $d\in\mathcal{D}$. If $\delta\colon d\to d'$ is a morphism in $\mathcal{D}$, we obtain a functor $\delta^*\colon d'/\varphi\to d/\varphi$ given by $(c',f')\mapsto(c',f'\delta)$. Observe that for all $e\in\mathcal{E}$ and $(c',f')\in d'/\varphi$ the diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{E}}(e,\overline{F}(d)) \arrow[r, equal] \arrow[d, "\overline{F}(\delta)_*"'] & \operatorname{Hom}_{\mathrm{Fun}(d/\varphi,\mathsf{Set})}(\mathrm{const}_e,Ft) \arrow[r, "\mathrm{pr}_{(c',f'\delta)}"] \arrow[d, "\delta^*"] & \operatorname{Hom}_{\mathcal{E}}(e,F(c')) \arrow[d, equal]\\
\operatorname{Hom}_{\mathcal{E}}(e,\overline{F}(d')) \arrow[r, equal] & \operatorname{Hom}_{\mathrm{Fun}(d'/\varphi,\mathsf{Set})}(\mathrm{const}_e,Ft\delta) \arrow[r, "\mathrm{pr}_{(c',f')}"'] & \operatorname{Hom}_{\mathcal{E}}(e,F(c'))
\end{tikzcd}$$

commutes. By the Yoneda lemma, we deduce that the left triangle

$$\begin{tikzcd}
\overline{F}(d) \arrow[d, "\overline{F}(\delta)"'] \arrow[dr, "\mathrm{pr}_{(c',f'\delta)}"] & \\
\overline{F}(d') \arrow[r, "\mathrm{pr}_{(c',f')}"'] & F(c')
\end{tikzcd}
\qquad
\begin{tikzcd}
\overline{F}(d) \arrow[r, "\mathrm{pr}_{(c,f)}"] \arrow[dr, "\mathrm{pr}_{(c,\varphi(h)f)}"'] & F(c) \arrow[d, "F(h)"]\\
& F(c')
\end{tikzcd}$$

commutes for all $\delta\colon d\to d'$ and $(c',f')\in d'/\varphi$, whereas the right triangle commutes by the definition of $\overline{F}$ as a limit, for all $(c,f)\in d/\varphi$ and $h\colon c\to c'$ in $\mathcal{C}$. In particular, $\overline{F}\colon\mathcal{D}\to\mathcal{E}$ defines a functor and the projection map $\mu_c = \mathrm{pr}_{(c,\mathrm{id}_{\varphi(c)})}\colon \overline{F}\varphi(c)\to F(c)$ is natural in c.

Now, for every functor $G\colon\mathcal{D}\to\mathcal{E}$ we need to show that the composite

$$\operatorname{Hom}_{\mathrm{Fun}(\mathcal{D},\mathcal{E})}(G,\overline{F}) \xrightarrow{\varphi^*} \operatorname{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(G\varphi,\overline{F}\varphi) \xrightarrow{\mu_*} \operatorname{Hom}_{\mathrm{Fun}(\mathcal{C},\mathcal{E})}(G\varphi,F) \tag{13.2}$$

is bijective. We first show injectivity. A natural transformation $\alpha\colon G\to\overline{F}$ consists of a map $\alpha_{d,c,f}\colon G(d)\to F(c)$ for every $d\in\mathcal{D}$ and $(c,f)\in d/\varphi$ satisfying the following conditions:

(a) for all maps $\delta\colon d \to d'$ and $(c, f') \in d'/\varphi$ the diagram

$$\begin{tikzcd} G(d) \arrow[r, "\alpha_{d,c,f'\delta}"] \arrow[d, "G(\delta)"'] & F(c) \\ G(d') \arrow[ur, "\alpha_{d',c,f'}"'] & \end{tikzcd}$$

commutes;

(b) for all $d \in \mathcal{D}$, all $(c, f) \in d/\varphi$ and all maps $h\colon c \to c'$ in $\mathcal{C}$ the diagram

$$\begin{tikzcd} G(d) \arrow[r, "\alpha_{d,c,f}"] \arrow[dr, "\alpha_{d,c',\varphi(h)f}"'] & F(c) \arrow[d, "F(h)"] \\ & F(c') \end{tikzcd}$$

commutes.

Indeed, the condition (b) shows that the $\alpha_{d,c,f}$, for varying (c, f), assemble into a map $\alpha_d\colon G(d) \to \overline{F}(d)$ such that $\alpha_{d,c,f} = \mathrm{pr}_{(c,f)} \circ \alpha_d$. Then the condition in (a) shows that the α_d are natural in d.

By (a) the diagram

$$\begin{tikzcd} G(d) \arrow[r, "\alpha_{d,c,f}"] \arrow[d, "G(f)"'] & F(c) \\ G\varphi(c) \arrow[ur, "\alpha_{\varphi(c),c,\mathrm{id}_{\varphi(c)}}"'] & \end{tikzcd}$$

commutes for all $(c, f) \in d/\varphi$. Hence, α is determined by $(\alpha_{\varphi(c),c,\mathrm{id}_{\varphi(c)}})_c$. By construction, the map (13.2) sends α to the natural transformation $(\alpha_{\varphi(c),c,\mathrm{id}_{\varphi(c)}})_c$, which shows that (13.2) is injective.

We now prove surjectivity: Let $\beta\colon G\varphi \to F$ be a natural transformation of functors $\mathcal{C} \to \mathcal{E}$. For every $d \in \mathcal{D}$ and $(c, f) \in d/\varphi$ we put

$$\alpha_{d,c,f} \coloneqq \beta_c \circ G(f)\colon G(d) \to F(c).$$

We need to check conditions (a) and (b). So let $\delta\colon d \to d'$ be a map in $\mathcal{D}$ and $(c, f') \in d'/\varphi$. Then we have a commutative diagram

$$\begin{tikzcd} G(d) \arrow[dr, "G(f'\delta)"] \arrow[drr, bend left, "\alpha_{d,c,f'\delta}"] \arrow[dd, "G(\delta)"'] & & \\ & G\varphi(c) \arrow[r, "\beta_c"] & F(c) \\ G(d') \arrow[ur, "G(f')"'] \arrow[urr, bend right, "\alpha_{d',c,f'}"'] & & \end{tikzcd}$$

which verifies (a). Finally, let $d \in \mathcal{D}$, $(c, f) \in d/\varphi$ and $h\colon c \to c'$ in $\mathcal{C}$. Then the diagram

$$\begin{tikzcd}
G(d) \arrow[r, "G(f)"] \arrow[d, equal] \arrow[rr, bend left, "\alpha_{d,c,f}"] & G\varphi(c) \arrow[r, "\beta_c"] \arrow[d, "G\varphi(h)"] & F(c) \arrow[d, "F(h)"] \\
G(d) \arrow[r, "G(\varphi(h)f)"'] \arrow[rr, bend right, "\alpha_{d,c',\varphi(h)f}"'] & G\varphi(c') \arrow[r, "\beta_{c'}"] & F(c'),
\end{tikzcd}$$

commutes, which verifies condition (b). It follows that (13.2) is surjective, hence bijective. □

Corollary 13.15. *Let $\varphi\colon \mathcal{C} \to \mathcal{D}$ be a functor such that $\mathcal{C}$ is small (and $\mathcal{D}$ is locally small). Let $\mathcal{E}$ be a complete (resp. cocomplete) category. Then every functor admits a right (resp. left) Kan extension along φ, which is pointwise.*

The following result explains why Kan extensions are called "extensions":

Corollary 13.16. *Let $\varphi\colon \mathcal{C} \hookrightarrow \mathcal{D}$ be a fully faithful functor. Let $F\colon \mathcal{C} \to \mathcal{E}$ be a functor and suppose that the left Kan extension $\varphi_! F\colon \mathcal{D} \to \mathcal{E}$ exists and is pointwise. Then the unit $\eta\colon F \xrightarrow{\sim} \varphi_! F \circ \varphi$ is an isomorphism of functors $\mathcal{C} \to \mathcal{E}$.*

Proof. By Proposition 13.14 we have a canonical isomorphism

$$\varphi_! F(\varphi(c)) \cong \operatorname{colim}\bigl(\varphi/\varphi(c) \xrightarrow{s} \mathcal{C} \xrightarrow{F} \mathcal{E}\bigr)$$

and $\eta_c\colon F(c) \to \operatorname{colim}_{(c',\varphi(c')\to\varphi(c))} F(c')$ is given by the canonical map corresponding to the element $(c, \mathrm{id}_{\varphi(c)}) \in \varphi/\varphi(c)$, which is a terminal object because φ is fully faithful. Hence η_c is an isomorphism. □

Proposition 13.17. *Consider the following diagram of functors*

$$\begin{tikzcd}
\mathcal{B} \arrow[r, "F"] \arrow[d, "\psi"'] & \mathcal{E} \\
\mathcal{C} \arrow[ur, dashed, "\psi_! F"] \arrow[d, "\varphi"'] & \\
\mathcal{D}. \arrow[uur, dashed] &
\end{tikzcd}$$

Suppose that the left Kan extension of F along ψ exists and is exhibited by the natural transformation $\eta\colon F \to \psi_! F \circ \psi$. Then the left Kan extension of $\psi_! F$ along φ exists if and only if the left Kan extension of F along $\varphi\psi$ exists, and in this case the canonical map $(\varphi\psi)_! F \xrightarrow{\sim} \varphi_!(\psi_! F)$ is an isomorphism. More precisely:

(a) If $\nu\colon \psi_! F \to \overline{F} \circ \varphi$ exhibits $\overline{F}\colon \mathcal{D} \to \mathcal{E}$ as the left Kan extension of $\psi_! F$ along φ, then

$$F \xrightarrow{\eta} (\psi_! F) \circ \psi \xrightarrow{\nu\psi} \overline{F} \circ \varphi\psi$$

exhibits $\overline{F}$ as the left Kan extension of F along $\varphi\psi$.

(b) If $\xi\colon F \to \overline{F} \circ \varphi\psi$ exhibits $\overline{F}\colon \mathcal{D} \to \mathcal{E}$ as the left Kan extension of F along $\varphi\psi$, then there exists a unique natural transformation $\nu\colon \psi_! F \to \overline{F} \circ \varphi$ satisfying $\xi = \nu\psi \circ \eta$, which exhibits $\overline{F}$ as the left Kan extension of $\psi_! F$ along φ.

The analogous result for right Kan extensions also holds.

Proof. Note that we have a commutative diagram of natural isomorphisms

$$\begin{tikzcd}
 & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(\psi_! F, G\varphi) \arrow[dr, "\sim"] & \\
\operatorname{Hom}_{\operatorname{Fun}(\mathcal{D},\mathcal{E})}(\overline{F}, G) \arrow[ur, dashed, "\sim"] \arrow[rr, dashed, "\sim"'] & & \operatorname{Hom}_{\operatorname{Fun}(\mathcal{B},\mathcal{E})}(F, G\varphi\psi)
\end{tikzcd}$$

where the right oblique arrow is induced by η.

In case (a) the left oblique arrow is induced by ν, and hence the composite is induced by $\nu\psi \circ \eta$.

In case (b) the horizontal arrow is induced by ξ. The left oblique isomorphism is then induced by the image of $\mathrm{id}_{\overline{F}}$, *i.e.*, the preimage of $\xi\colon F \to \overline{F}\varphi\psi$, in $\operatorname{Hom}_{\operatorname{Fun}(\mathcal{C},\mathcal{E})}(\psi_! F, \overline{F}\varphi)$. But by definition of $\psi_! F$, this is the unique map $\nu\colon \psi_! F \to \overline{F}\varphi$ such that $\nu\psi \circ \eta = \xi$. $\square$

§14. Kan extensions in triangulated categories

Proposition 14.1. *Let $(Q,\xi)\colon (\mathcal{C}, T_{\mathcal{C}}) \to (\mathcal{D}, T_{\mathcal{D}})$ and $(F,\mu)\colon (\mathcal{C}, T_{\mathcal{C}}) \to (\mathcal{E}, \mathcal{T}_{\mathcal{E}})$ be functors of categories with translation, where $\xi\colon QT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{D}}Q$ and $\mu\colon FT_{\mathcal{C}} \xrightarrow{\sim} T_{\mathcal{E}}F$ are natural isomorphisms. Suppose that the left Kan extension of F along Q exists.*

Then there exists a unique natural isomorphism $\nu\colon (Q_! F)T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{E}}(Q_! F)$ such that $(Q_! F, \nu)$ is a left adjoint object of (F, μ) under the functor

$$Q^*\colon \operatorname{Fun}\big((\mathcal{D}, T_{\mathcal{D}}), (\mathcal{E}, T_{\mathcal{E}})\big) \longrightarrow \operatorname{Fun}\big((\mathcal{C}, T_{\mathcal{C}}), (\mathcal{E}, T_{\mathcal{E}})\big)$$

In other words: Let $\eta\colon F \to (Q_! F)Q$ be the natural transformation exhibiting $Q_! F$ as the left Kan extension of F along Q. Then:

(i) There exists a unique natural isomorphism $\nu\colon (Q_! F)T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{E}}(Q_! F)$ such that

$$\eta\colon (F,\mu) \longrightarrow (Q_! F, \nu) \circ (Q, \xi)$$

is a natural transformation of functors with translation.

(ii) For every functor $(G,\rho)\colon (\mathcal{D}, T_{\mathcal{D}}) \to (\mathcal{E}, T_{\mathcal{E}})$ the natural map

$$\operatorname{Hom}\big((Q_! F, \nu), (G,\rho)\big) \xrightarrow{\sim} \operatorname{Hom}\big((F,\mu), (G,\rho) \circ (Q,\xi)\big)$$

induced by η is bijective.

A similar result holds for right Kan extensions.

Proof. We first prove (i). Unraveling the definitions, we need to show that there exists a unique isomorphism $\nu\colon (Q_!F)T_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{E}}(Q_!F)$ making the following diagram commutative:

$$\begin{tikzcd}
FT_{\mathcal{C}} \arrow[rr, "\mu", "\sim"'] \arrow[d, "\eta T_{\mathcal{C}}"'] & & T_{\mathcal{E}}F \arrow[d, "T_{\mathcal{E}}\eta"] \\
(Q_!F)QT_{\mathcal{C}} \arrow[r, "\sim", "(Q_!F)\xi"'] & (Q_!F)T_{\mathcal{D}}Q \arrow[r, "\sim", "\nu Q"'] & T_{\mathcal{E}}(Q_!F)Q.
\end{tikzcd}$$

This statement follows from the claim that the composite

$$FT_{\mathcal{C}} \xrightarrow{\eta T_{\mathcal{C}}} (Q_!F)QT_{\mathcal{C}} \xrightarrow{(Q_!F)\xi} (Q_!F)T_{\mathcal{D}}Q \tag{14.1}$$

exhibits $(Q_!F)T_{\mathcal{D}}$ as the left Kan extension of $FT_{\mathcal{C}}$ along Q. We consider the following diagram

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dl, "Q"'] \arrow[dr, "T_{\mathcal{C}}^{-1}"] \arrow[rr, bend left, "F"] & & \\
\mathcal{D} \arrow[dr, "T_{\mathcal{D}}^{-1}"'] & & \mathcal{C} \arrow[r, "FT_{\mathcal{C}}"] \arrow[dl, "Q"] & \mathcal{E} \\
 & \mathcal{D} \arrow[urr, dashed, "(Q_!F)T_{\mathcal{D}}=Q_!(FT_{\mathcal{C}})"'] & &
\end{tikzcd}$$

By Observation 7.1 the functor $T_{\mathcal{D}}$ is right adjoint to $T_{\mathcal{D}}^{-1}$, so the unit $\alpha_{\mathcal{D}}\colon \mathrm{id}_{\mathcal{D}} \xrightarrow{\sim} T_{\mathcal{D}}T_{\mathcal{D}}^{-1}$ and counit $\beta_{\mathcal{D}}T_{\mathcal{D}}^{-1}T_{\mathcal{D}} \xrightarrow{\sim} \mathrm{id}_{\mathcal{D}}$ are isomorphisms. Similarly, for $T_{\mathcal{C}}^{-1} \dashv T_{\mathcal{C}}$ with unit $\alpha_{\mathcal{C}}$. We consider the following commutative diagram

$$\begin{tikzcd}
F \arrow[r, "\eta"] \arrow[d, "F\alpha_{\mathcal{C}}"'] & (Q_!F)Q \arrow[r, "Q_!F\alpha_{\mathcal{D}}Q"] \arrow[d, "(Q_!F)Q\alpha_{\mathcal{C}}"] & (Q_!F)T_{\mathcal{D}} \circ T_{\mathcal{D}}^{-1}Q \arrow[d, "(Q_!F)T_{\mathcal{D}}\xi'"] \\
FT_{\mathcal{C}} \circ T_{\mathcal{C}}^{-1} \arrow[r, "\eta T_{\mathcal{C}}T_{\mathcal{C}}^{-1}"'] & (Q_!F)QT_{\mathcal{C}}T_{\mathcal{C}}^{-1} \arrow[r, "(Q_!F)\xi T_{\mathcal{C}}^{-1}"'] & (Q_!F)T_{\mathcal{D}} \circ QT_{\mathcal{C}}^{-1},
\end{tikzcd} \tag{14.2}$$

where $\xi'\colon T_{\mathcal{D}}^{-1}Q \xrightarrow{\sim} QT_{\mathcal{C}}^{-1}$ is defined as the mate of ξ, so that the diagram

$$\begin{tikzcd}
Q \arrow[r, "Q\alpha_{\mathcal{C}}"] \arrow[d, "\alpha_{\mathcal{D}}Q"'] & QT_{\mathcal{C}}T_{\mathcal{C}}^{-1} \arrow[r, "\xi T_{\mathcal{C}}^{-1}"] \arrow[d, "\alpha_{\mathcal{D}}QT_{\mathcal{C}}T_{\mathcal{C}}^{-1}"'] & T_{\mathcal{D}}QT_{\mathcal{C}}^{-1} \arrow[d, "\alpha_{\mathcal{D}}T_{\mathcal{D}}QT_{\mathcal{C}}^{-1}"'] \arrow[dr, equal] & \\
T_{\mathcal{D}}T_{\mathcal{D}}^{-1}Q \arrow[r, "T_{\mathcal{D}}T_{\mathcal{D}}^{-1}Q\alpha_{\mathcal{C}}"'] \arrow[rrr, bend right, "T_{\mathcal{D}}\xi'"'] & T_{\mathcal{D}}T_{\mathcal{D}}^{-1}QT_{\mathcal{C}}T_{\mathcal{C}}^{-1} \arrow[r, "T_{\mathcal{D}}T_{\mathcal{D}}^{-1}\xi T_{\mathcal{C}}^{-1}"'] & T_{\mathcal{D}}T_{\mathcal{D}}^{-1}T_{\mathcal{D}}QT_{\mathcal{C}}^{-1} \arrow[r, "T_{\mathcal{D}}\beta_{\mathcal{D}}QT_{\mathcal{C}}^{-1}"'] & T_{\mathcal{D}}QT_{\mathcal{C}}^{-1}
\end{tikzcd}$$

commutes.

The upper-right circuit in (14.2) shows that $(Q_!F)T_{\mathcal{D}}$ is the left Kan extension of F along $QT_{\mathcal{C}}^{-1}$ (combine Proposition 13.17 and Proposition 13.10 (and also Exercise 13.5)). The lower-right circuit then shows that this left Kan extension is exhibited by the natural transformation

$$F \xrightarrow{F\alpha_{\mathcal{C}}} FT_{\mathcal{C}}T_{\mathcal{C}}^{-1} \xrightarrow{[(Q_!F)\xi\circ\eta T_{\mathcal{C}}]T_{\mathcal{C}}^{-1}} (Q_!F)T_{\mathcal{D}} \circ QT_{\mathcal{C}}^{-1}.$$

Again by Proposition 13.17 we deduce that (14.1) exhibits $(Q_!F)T_{\mathcal{D}}$ as the left Kan extension of $FT_{\mathcal{C}}$ along Q. This finishes the proof of (i).

We now prove (ii). By (i) we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}\big((Q_!F,\nu),(G,\rho)\big) \arrow[r] \arrow[d, hook] & \mathrm{Hom}\big((F,\mu),(G,\rho)\circ(Q,\xi)\big) \arrow[d, hook] \\
\mathrm{Hom}(Q_!F,G) \arrow[r, "\sim"] & \mathrm{Hom}(F,G\circ Q).
\end{tikzcd}$$

We immediately deduce that the top horizontal map is injective. We now prove surjectivity, so let $\alpha \in \mathrm{Hom}((F,\mu),(G,\rho)\circ(Q,\xi))$. Since the bottom map is surjective, there exists $\beta\colon Q_!F \to G$ such that $\alpha = \beta Q \circ \eta$. Since α is compatible with the translations, we deduce that the outer diagram

$$\begin{tikzcd}
FT_{\mathcal{C}} \arrow[rr, "\mu"] \arrow[d, "\eta T_{\mathcal{C}}"'] \arrow[dd, bend right=60, "\alpha T_{\mathcal{C}}"'] & & T_{\mathcal{E}}F \arrow[d, "T_{\mathcal{E}}\eta"] \arrow[dd, bend left=60, "T_{\mathcal{E}}\alpha"] \\
(Q_!F)QT_{\mathcal{C}} \arrow[r, "(Q_!F)\xi"] \arrow[d, "\beta QT_{\mathcal{C}}"'] & (Q_!F)T_{\mathcal{D}}Q \arrow[r, "\nu Q"] \arrow[d, "\beta T_{\mathcal{D}}Q"] & T_{\mathcal{E}}(Q_!F)Q \arrow[d, "T_{\mathcal{E}}\beta Q"] \\
GQT_{\mathcal{C}} \arrow[r, "G\xi"'] & GT_{\mathcal{D}}Q \arrow[r, "\rho Q"'] & T_{\mathcal{E}}GQ
\end{tikzcd}$$

is commutative. Now the top rectangle commutes by (i) and the lower left square commutes by naturality. Again by the proof of (i) the composite $FT_{\mathcal{C}} \xrightarrow{\eta T_{\mathcal{C}}} (Q_!F)QT_{\mathcal{C}} \xrightarrow{(Q_!F)\xi} (Q_!F)T_{\mathcal{D}}Q$ exhibits $(Q_!F)T_{\mathcal{D}}$ as the left Kan extension of $FT_{\mathcal{C}}$ along Q, *i.e.*, the map

$$\mathrm{Hom}((Q_!F)T_{\mathcal{D}}, T_{\mathcal{E}}G) \xrightarrow{\sim} \mathrm{Hom}(FT_{\mathcal{C}}, T_{\mathcal{E}}GQ)$$

is an isomorphism. But under this map both $T_{\mathcal{E}}\beta\circ\nu$ and $\rho\circ\beta T_{\mathcal{D}}$ are sent to the same natural transformation $FT_{\mathcal{C}} \to T_{\mathcal{E}}GQ$, hence they are equal. But this shows that β lies in $\mathrm{Hom}((Q_!F,\nu),(G,\rho))$ as desired. $\square$

Definition 14.2. Let $(\mathcal{C},T)$ be a triangulated category and let $\mathcal{N}\subseteq\mathcal{C}$ be a triangulated subcategory. Denote $(Q,\xi)\colon(\mathcal{C},T)\to(\mathcal{C}/\mathcal{N},T')$ the localization functor. Let $(F,\mu)\colon(\mathcal{C},T)\to(\mathcal{D},S)$ be an exact functor.

(a) A left Kan extension $Q_!F\colon\mathcal{C}/\mathcal{N}\to\mathcal{D}$ of F along Q is called *exact* if $(Q_!F,\mu')$ is exact, where $\mu'\colon Q_!F\circ T' \xrightarrow{\sim} S\circ Q_!F$ is the natural isomorphism from Proposition 14.1.

(b) A right Kan extension $Q_*F\colon\mathcal{C}/\mathcal{N}\to\mathcal{D}$ of F along Q is called *exact* if (Q_*F,μ'') is exact, where $\mu''\colon Q_*F\circ T' \xrightarrow{\sim} S\circ Q_*F$ is the natural isomorphism from Proposition 14.1.

Remark 14.3. Suppose that $(Q,\xi)\colon(\mathcal{C},T)\to(\mathcal{C}/\mathcal{N},T')$ is a localization and $(F,\mu)\colon(\mathcal{C},T)\to(\mathcal{D},S)$ is an exact functor such that the left Kan extension $Q_!F\colon\mathcal{C}/\mathcal{N}\to\mathcal{D}$ is absolute.

Deligne proves in [Del06, Proposition 1.2.2(ii)] that the functor $(Q_!F,\nu)$ is exact, where ν is the natural isomorphism from Proposition 14.1. An accessible explanation can also be found at [GVu], but the proof is outside the scope of this lecture.

Definition 14.4. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Let $* \in \{\varnothing, +, -, b\}$ and denote by $Q_{\mathcal{A}}\colon \mathsf{K}^*(\mathcal{A}) \to \mathsf{D}^*(\mathcal{A})$ and $Q_{\mathcal{B}}\colon \mathsf{K}(\mathcal{B}) \to \mathsf{D}(\mathcal{B})$ the localization functors at the quasi-isomorphisms.

$$\begin{tikzcd} \mathsf{K}^*(\mathcal{A}) \arrow[r, "\mathsf{K}F"] \arrow[d, "Q_{\mathcal{A}}"'] \arrow[dr, phantom, "\swarrow\eta"] & \mathsf{K}(\mathcal{B}) \arrow[d, "Q_{\mathcal{B}}"] \\ \mathsf{D}^*(\mathcal{A}) \arrow[r, dashed, "\mathrm{R}F"'] & \mathsf{D}(\mathcal{B}). \end{tikzcd}$$

Then an exact absolute left Kan extension

$$\mathrm{R}F \coloneqq Q_{\mathcal{A}!}(Q_{\mathcal{B}} \circ \mathsf{K}F)\colon \mathsf{D}^*(\mathcal{A}) \longrightarrow \mathsf{D}(\mathcal{B})$$

(if it exists) is called the *right derived functor* of F. Note that $\mathrm{R}F$ comes with an exact natural transformation $\eta\colon Q_{\mathcal{B}} \circ \mathsf{K}F \to \mathrm{R}F \circ Q_{\mathcal{A}}$ and satisfies the following universal property: For all exact functors $(G, \rho)\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ and natural transformations $\zeta\colon Q_{\mathcal{B}} \circ \mathsf{K}F \to G \circ Q_{\mathcal{A}}$ there exists a unique natural transformation $\xi\colon \mathrm{R}F \to G$ such that $\zeta = \xi Q_{\mathcal{A}} \circ \eta$:

$$\begin{tikzcd} Q_{\mathcal{B}}\mathsf{K}F \arrow[r, "\eta"] \arrow[dr, "\zeta"'] & \mathrm{R}F \circ Q_{\mathcal{A}} \arrow[d, "\xi Q_{\mathcal{A}}"] & & \mathrm{R}F \arrow[d, dashed, "\exists!\xi"] \\ & G \circ Q_{\mathcal{A}} & & G. \end{tikzcd}$$

We denote by R^iF (for $i \in \mathbb{Z}$) the composition

$$\mathcal{A} \hookrightarrow \mathsf{D}^*(\mathcal{A}) \xrightarrow{\mathrm{R}F} \mathsf{D}(\mathcal{B}) \xrightarrow{\mathrm{H}^i} \mathcal{B},$$

and call it the *i-th right derived functor of F*.

Example 14.5. Let $F\colon \mathcal{A} \to \mathcal{B}$ be an exact functor of abelian categories. Then the right derived functor $\mathrm{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists and is given by applying F componentwise to a complex. This follows from Lemma 12.9 and Lemma 13.8.

Remark 14.6 (Relation to universal δ-functors)**.** Let $F\colon \mathcal{A} \to \mathcal{B}$ be a (left exact) functor and let $\mathrm{R}F\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ be the right derived functor of F.

For every short exact sequence $0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$ we obtain a distinguished triangle $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{\delta} A[1]$ and hence a long exact sequence

$$\begin{tikzcd} \cdots \arrow[r] & \mathrm{R}^0F(A) \arrow[r, "\mathrm{R}^0F(f)"] & \mathrm{R}^0F(B) \arrow[r, "\mathrm{R}^0F(g)"] & \mathrm{R}^0F(C) \arrow[dll, rounded corners, "\delta^0"'] \\ & \mathrm{R}^1F(A) \arrow[r, "\mathrm{R}^1F(f)"] & \mathrm{R}^1F(B) \arrow[r, "\mathrm{R}^1F(g)"] & \mathrm{R}^1F(C) \arrow[dll, rounded corners, "\delta^1"'] \\ & \mathrm{R}^2F(A) \arrow[r] & \cdots \end{tikzcd}$$

which is visibly natural in the short exact sequence $0 \to A \to B \to C \to 0$. Hence, $(\{\mathrm{R}^iF\}_i, \{\delta^i\}_i)$ defines a δ-functor. Moreover, the natural transformation $\eta\colon Q_{\mathcal{B}}\mathsf{K}F \to \mathrm{R}FQ_{\mathcal{A}}$ induces a natural transformation $\eta'\colon F \to \mathrm{R}^0F$ of functors $\mathcal{A} \to \mathcal{B}$.

At this point, we may ask whether it is true that $(\{\mathrm{R}^i F\}_i, \{\delta^i\}_i)$ is universal. It seems that this question cannot be resolved from the general theory alone. We will next prove a criterion for the existence of right derived functors, which allows us to give an affirmative answer.

Remark 14.7. Let $\mathrm{R}F\colon \mathcal{D}^*(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ be the right derived functor of an additive functor $F\colon \mathcal{A} \to \mathcal{B}$ between abelian categories. Then

(i) $\mathrm{R}^i F = 0$ for all $i < 0$.

(ii) $\mathrm{R}^0 F\colon \mathcal{A} \to \mathcal{B}$ is left exact.

Proof. (ii) follows from (i) and the long exact sequence. For the proof of (i) we start with a general fact about colimits:

Step 1: Every right adjoint functor $R\colon \mathcal{J} \to \mathcal{I}$ is *cofinal*, *i.e.*, for every functor $F\colon \mathcal{I} \to \mathcal{C}$ the colimit $\operatorname{colim} F$ exist if and only if $\operatorname{colim} FR$ exists and in this case the canonical map

$$\operatorname{colim}\bigl(\mathcal{J} \xrightarrow{R} \mathcal{I} \xrightarrow{F} \mathcal{C}\bigr) \xrightarrow{\sim} \operatorname{colim}\bigl(\mathcal{I} \xrightarrow{F} \mathcal{C}\bigr)$$

is an isomorphism.

Let $p_{\mathcal{I}}\colon \mathcal{I} \to *$ and $p_{\mathcal{J}}\colon \mathcal{J} \to *$ be the projections and denote by $L\colon \mathcal{I} \to \mathcal{J}$ the left adjoint of R. For any $c \in \mathcal{C}$, viewed as a functor $* \to \mathcal{C}$ we have that $cp_{\mathcal{J}}L = cp_{\mathcal{I}}$ and $cp_{\mathcal{J}} = cp_{\mathcal{J}}LR$ are the constant functors with value c. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, cp_{\mathcal{J}}L) \arrow[r, "\sim"] \arrow[d, equal] & \operatorname{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(FR, cp_{\mathcal{J}}) \arrow[d, equal] \\
\operatorname{Hom}_{\mathrm{Fun}(\mathcal{I},\mathcal{C})}(F, cp_{\mathcal{I}}) \arrow[r, "R", "\sim"'] \arrow[d, equal] & \operatorname{Hom}_{\mathrm{Fun}(\mathcal{J},\mathcal{C})}(FR, cp_{\mathcal{J}}) \arrow[d, equal] \\
\operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} F, c) \arrow[r, dashed] & \operatorname{Hom}_{\mathcal{C}}(\operatorname{colim} FR, c),
\end{tikzcd}$$

where the top horizontal map is an isomorphism by the adjunction. Hence the middle horizontal map is an isomorphism, which proves the claim.

Step 2: Denote by $\mathsf{K}^{\geq 0}(\mathcal{A}) \subseteq \mathsf{K}^*(\mathcal{A})$ and $\mathsf{D}^{\geq 0}(\mathcal{A}) \subseteq \mathsf{D}^*(\mathcal{A})$ the full subcategories spanned by the complexes X with $\mathrm{H}^i(X) = 0$ for all $i < 0$, and let $Q^{\geq 0}\colon \mathsf{K}^{\geq 0}(\mathcal{A}) \to \mathsf{D}^{\geq 0}(\mathcal{A})$ be the functor induced by Q. For every $X \in \mathsf{D}^{\geq 0}(\mathcal{A})$ the inclusion functor

$$Q^{\geq 0}/X \longrightarrow Q/X$$

is a right adjoint and hence cofinal by Step 1. Note that every map $f\colon Q(Y) \to X$ factors uniquely as a composite $Q(Y) \to \tau^{\geq 0}Q(Y) = Q^{\geq 0}(\tau^{\geq 0}Y) \xrightarrow{f'} X$, and hence one easily checks that the left adjoint is given by sending $(Y, f) \mapsto (\tau^{\geq 0}Y, f')$.

Step 3: We now prove the claim. Note that $\mathrm{R}F\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ is the *absolute* left Kan extension of $Q_{\mathcal{B}} \circ \mathsf{K}F$ along $Q_{\mathcal{A}}$. Hence, for any $i < 0$, $\mathrm{H}^i \circ \mathrm{R}F$ is the left Kan extension of $\mathrm{H}^i\mathsf{K}F$ along $Q_{\mathcal{A}}$. Note that $\mathrm{H}^i\mathrm{R}F$ is again absolute and in particular pointwise. By Proposition 13.14 and Steps 1 and 2 we compute, for all $X \in \mathsf{D}^{\geq 0}(\mathcal{A})$,

$$\mathrm{H}^i\mathrm{R}F(X) = \operatorname*{colim}_{Y \in Q/X} \mathrm{H}^i\mathsf{K}F(Y) \xleftarrow{\sim} \operatorname*{colim}_{Z \in Q^{\geq 0}/X} \mathrm{H}^i\mathsf{K}^{\geq 0}F(Z) = 0.$$

$\square$

§15. Existence of derived functors

We start with a general existence criterion.

Theorem 15.1. *Let $(\mathcal{C}, T)$ be a triangulated category and $\mathcal{N} \subseteq \mathcal{C}$ a triangulated subcategory, and denote by $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ the localization functor. Let $F\colon \mathcal{C} \to \mathcal{D}$ be an exact functor and suppose that there exists a full triangulated subcategory $\iota\colon \mathcal{L} \hookrightarrow \mathcal{C}$ satisfying the following conditions:*

- *For every $X \in \mathcal{C}$ there exists a map $s\colon X \to \iota Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{L}$ (that is, there exists a distinguished triangle $X \xrightarrow{s} \iota Y \to N \to T(X)$ with $N \in \mathcal{N}$).*
- *We have $F(Y) = 0$ for all $Y \in \mathcal{L} \cap \mathcal{N}$.*

Then there exists an exact, absolute left Kan extension $Q_!F\colon \mathcal{C}/\mathcal{N} \to \mathcal{D}$ of F along Q. Moreover, for every $Y \in \mathcal{L}$ the canonical map

$$F(Y) \xrightarrow{\sim} Q_!F(QY)$$

is an isomorphism.

Proof. Let $\iota\colon \mathcal{L} \hookrightarrow \mathcal{C}$ be the inclusion and put $\mathcal{N}' \coloneqq \mathcal{L} \cap \mathcal{N}$. Observe that the induced functor $j\colon \mathcal{L}/\mathcal{N}' \xrightarrow{\sim} \mathcal{C}/\mathcal{N}$ is an equivalence of categories by Corollary 10.9. Consider the following commutative diagram:

$$\begin{tikzcd}
 & \mathcal{C} \arrow[dr, "Q"] \arrow[rr, bend left, "F"] & & \\
\mathcal{L} \arrow[ur, hook, "\iota"] \arrow[dr, "Q_{\mathcal{L}}"'] & & \mathcal{C}/\mathcal{N} \arrow[r, dashed] & \mathcal{D} \\
 & \mathcal{L}/\mathcal{N}' \arrow[ur, "j", "\sim"'] \arrow[urr, bend right, "Q_{\mathcal{L}!}(F\iota)"'] & &
\end{tikzcd}$$

For every (not necessarily exact) functor $G\colon \mathcal{C}/\mathcal{N} \to \mathcal{D}$ we have the following natural maps

$$\begin{aligned}\operatorname{Hom}(F, GQ) \xrightarrow{\iota^*} \operatorname{Hom}(F\iota, GQ\iota) &\cong \operatorname{Hom}(F\iota, GjQ_{\mathcal{L}}) \\ &\cong \operatorname{Hom}(Q_{\mathcal{L}!}(F\iota), Gj) \cong \operatorname{Hom}\big(Q_{\mathcal{L}!}(F\iota)j^{-1}, G\big).\end{aligned}$$

We claim that ι^* is bijective. Once this is proven, it follows that $Q_{\mathcal{L}!}(\iota F)j^{-1}\colon \mathcal{C}/\mathcal{N} \to \mathcal{D}$ is the left Kan extension of F along Q. Since the outer diagram commutes, we compute

$$F(\iota Y) = Q_{\mathcal{L}!}(F\iota)Q_{\mathcal{L}}(Y) = Q_{\mathcal{L}!}(F\iota)j^{-1}jQ_{\mathcal{L}}(Y) = Q_!F(Q\iota Y),$$

for every $Y \in \mathcal{L}$, which implies the last assertion of the theorem. It follows from Lemma 13.8 and Theorem 11.2 that $Q_{\mathcal{L}!}(F\iota)$, hence also $Q_!F$, is absolute and exact.

It remains to prove that $\iota^*\colon \operatorname{Hom}(F, GQ) \to \operatorname{Hom}(F\iota, GQ\iota)$ is bijective. Note that the functor $H \coloneqq GQ\colon \mathcal{C} \to \mathcal{D}$ has the following property: For all $X \in \mathcal{C}$ there exists a map $s\colon X \to \iota Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{L}$ such that:

(a) $H(s)$ is an isomorphism, and

(b) For all maps $t\colon X \to \iota Y'$ with $Y' \in \mathcal{L}$ (not necessarily in $S_{\mathcal{N}}$) there exist maps $Y \xrightarrow{t'} Y'' \xleftarrow{s'} Y'$ in $\mathcal{L}$ such that $H\iota(s')$ is an isomorphism and the diagram

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & \iota Y \arrow[d, "\iota t'"] \\ \iota Y' \arrow[r, "\iota s'"'] & \iota Y'' \end{tikzcd}$$

commutes.

Indeed, for (a) this is obvious. Let us verify (b). By Theorem 11.3(i) and (S3) we can complete the diagram $\iota Y' \xleftarrow{t} X \xrightarrow{s} \iota Y$ to a commutative square

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & \iota Y \arrow[d, "\tilde{t}"] \\ \iota Y' \arrow[r, "\tilde{s}"'] & Z \end{tikzcd}$$

with $\tilde{s} \in S_{\mathcal{N}}$. By assumption, we find a map $\tilde{s}'\colon Z \to \iota Y''$ in $S_{\mathcal{N}}$ with $Y'' \in \mathcal{L}$. Since ι is fully faithful, there exist unique maps $s'\colon Y' \to Y''$ and $t'\colon Y \to Y''$ such that $\iota(s') = \tilde{s}'\tilde{s}$ and $\iota(t') = \tilde{s}'\tilde{t}$ as desired.

We first prove that $\iota^*\colon \operatorname{Hom}(F, H) \to \operatorname{Hom}(F\iota, H\iota)$ is injective. To this end, let $\alpha, \beta\colon F \to H$ be natural transformations such that $\alpha\iota = \beta\iota$. Let now $X \in \mathcal{C}$ be fixed but arbitrary. By assumption we find a map $s\colon X \to \iota Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{L}$ such that $H(s)$ is an isomorphism. We calculate

$$H(s)\alpha_X = \alpha_{\iota Y} F(s) = \beta_{\iota Y} F(s) = H(s)\beta_X.$$

As $H(s)$ is an isomorphism, we deduce $\alpha_X = \beta_X$. Since X was arbitrary, it follows that $\alpha = \beta$.

We now prove that ι^* is surjective. Let $\alpha\colon F\iota \to H\iota$ be a natural transformation. We define a new natural transformation $\beta\colon F \to H$ as follows: For every $X \in \mathcal{C}$ we choose a map $s\colon X \to \iota Y$ in $S_{\mathcal{N}}$ with $Y \in \mathcal{L}$ and define $\beta_X \coloneqq H(s)^{-1}\alpha_Y F(s)\colon F(X) \to H(X)$. We need to check that β_X is independent of the choice of s and defines a natural transformation. Let $t\colon X \to \iota Y'$ be any map and choose $Y \xrightarrow{t'} Y'' \xleftarrow{s'} Y'$ as in (b). Consider the following diagram:

$$\begin{tikzcd} F(X) \arrow[rrr, dashed, "\beta_X"] \arrow[dr, "F(s)"] \arrow[ddd, "F(t)"'] & & & H(X) \arrow[dl, "\sim"', "H(s)"] \arrow[ddd, "H(t)"] \\ & F(\iota Y) \arrow[r, "\alpha_Y"] \arrow[d, "F(\iota t')"'] & H(\iota Y) \arrow[d, "H(\iota t')"] & \\ & F(\iota Y'') \arrow[r, "\alpha_{Y''}"'] & H(\iota Y'') & \\ F(\iota Y') \arrow[ur, "F(\iota s')"] \arrow[rrr, "\alpha_{Y'}"'] & & & H(\iota Y'). \arrow[ul, "\sim", "H(\iota s')"'] \end{tikzcd}$$

Since every small quadrilateral commutes and $H(\iota s')$ is an isomorphism, we deduce that the outer square commutes. In particular, β_X does not depend on the choice of s. Let now $f\colon X \to X'$ be a morphism in $\mathcal{C}$. Choose maps $s\colon X \to \iota Y$ and $t\colon X' \to \iota Y'$ in $S_{\mathcal{N}}$ with $Y, Y' \in \mathcal{L}$. By (b) we find maps $Y \xrightarrow{t'} Y'' \xleftarrow{s'} Y'$ in $\mathcal{L}$ such that $\iota s' \circ tf = \iota t' \circ s$. Consider the diagram

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d, "\beta_X"'] & F(X') \arrow[r, "F(t)"] \arrow[d, "\beta_{X'}"] & F(\iota Y') \arrow[d, "\alpha_{Y'}"] \\ H(X) \arrow[r, "H(f)"'] & H(X') \arrow[r, "\sim", "H(t)"'] & H(\iota Y'), \end{tikzcd}$$

where the outer diagram and the right square commute by what we have shown above. Since $H(t)$ is an isomorphism, it follows that the left square commutes, *i.e.*, $\beta\colon F \to H$ is a natural transformation such that $\beta\iota = \alpha$. Hence ι^* is surjective. This finishes the proof. $\square$

Theorem 15.2. *Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Suppose there is a full subcategory $\mathcal{L} \subseteq \mathcal{A}$ satisfying the following properties:*

(a) For every $X \in \mathcal{A}$ there exists an injection $X \hookrightarrow L$ into some $L \in \mathcal{L}$.

(b) Let $0 \to L \to A \to B \to 0$ be a short exact sequence with $L \in \mathcal{L}$. Then we have $A \in \mathcal{L}$ if and only if $B \in \mathcal{L}$.

Then:

(i) Every complex $X \in \mathsf{K}^+(\mathcal{A})$ admits a quasi-isomorphism $X \to L$ into a complex $L \in \mathsf{K}^+(\mathcal{L})$.

(ii) Suppose that the following condition is satisfied:

(c) For all short exact sequences $0 \to L' \to L \to L'' \to 0$ with $L', L, L'' \in \mathcal{L}$, the induced sequence $0 \to F(L') \to F(L) \to F(L'') \to 0$ is exact.

The right derived functor $\mathrm{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists and can be computed as

$$\mathrm{R}F(Q_{\mathcal{A}}(X)) = Q_{\mathcal{B}}\mathsf{K}F(L),$$

where $X \to L$ is any quasi-isomorphism with $L \in \mathsf{K}^+(\mathcal{L})$.

Proof. Let $X \in \mathsf{K}^+(\mathcal{A})$ and fix i_0 with $X^i = 0$ for all $i < i_0$. Put $L^i = 0$ for all $i < i_0$. Assume now that for some $n \in \mathbb{Z}$ we have constructed a morphism of complexes

$$\begin{tikzcd} X\colon \arrow[d, "f"'] & \cdots \arrow[r] & X^{n-2} \arrow[r, "d_X^{n-2}"] \arrow[d, "f^{n-2}"'] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] & X^n \arrow[r] & \cdots \\ L^{<n}\colon & \cdots \arrow[r, "d_L^{n-3}"'] & L^{n-2} \arrow[r, "d_L^{n-2}"'] & L^{n-1} & & \end{tikzcd}$$

satisfying the following property

($*_n$) The induced morphism $\mathrm{H}^i(X) \to \mathrm{H}^i(L^{<n})$ is an isomorphism for $i \leq n-2$ and a monomorphism for $i = n-1$.

We now define M^n and N^n in $\mathcal{A}$ by the following pushout squares:

$$\begin{tikzcd}
\operatorname{Coker}(d_X^{n-2}) \arrow[r] \arrow[d] & \operatorname{Ker}(d_X^n) \arrow[r, hook] \arrow[d] & X^n \arrow[d] \\
\operatorname{Coker}(d_L^{n-2}) \arrow[r] & M^n \arrow[r, hook] \arrow[ul, phantom, "\ulcorner", very near start] & N^n. \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

Note that, since $\operatorname{Ker}(d_X^n) \hookrightarrow X^n$ is monic, Proposition 3.10 shows that the right square is also a pullback and $M^n \hookrightarrow N^n$ is monic. Choose a monomorphism $N^n \hookrightarrow L^n$ with $L^n \in \mathcal{L}$ and let $d_L^{n-1} \colon L^{n-1} \to L^n$ and $f^n \colon X^n \to L^n$ be the obvious maps. From the construction it is clear that $f^n d_X^{n-1} = d_L^{n-1} f^{n-1}$. Hence, we obtain a morphism $X \to L^{\leq n}$ of complexes, and it remains to check that the property $(*_{n+1})$ is satisfied. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^{n-1}(X) \arrow[r] \arrow[d, "\sim"] & \operatorname{Coker}(d_X^{n-2}) \arrow[r, "d'_X"] \arrow[d] & \operatorname{Ker}(d_X^n) \arrow[r] \arrow[d] & \mathrm{H}^n(X) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \operatorname{Ker}(d'_L) \arrow[r] & \operatorname{Coker}(d_L^{n-2}) \arrow[r, "d'_L"'] & M^n \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & \mathrm{H}^n(X) \arrow[r] & 0.
\end{tikzcd}$$

The top row is exact by construction. For the exactness of the bottom row, we note that Proposition 3.10 shows that $\mathrm{H}^n(X) = \operatorname{Coker}(d'_X) \xrightarrow{\sim} \operatorname{Coker}(d'_L)$ is an isomorphism. Note also that $\mathrm{H}^{n-1}(X) \xrightarrow{\sim} \operatorname{Ker}(d'_L)$ is an isomorphism. It is epic by Proposition 3.10, and it is monic by the assumption that $\mathrm{H}^{n-1}(X) \hookrightarrow \mathrm{H}^{n-1}(L^{<n})$ is monic.

As $M^n \hookrightarrow L^n$ is a monomorphism, we finally conclude that $\mathrm{H}^{n-1}(X) \xrightarrow{\sim} \mathrm{H}^{n-1}(L^{\leq n})$ is an isomorphism and $\mathrm{H}^n(X) \hookrightarrow \mathrm{H}^n(L^{\leq n})$ is monic. This finishes the induction step and hence part (i).

We now prove (ii). By (b), $\mathcal{L}$ is closed under direct sums. Therefore, $\mathsf{K}^+(\mathcal{L}) \subseteq \mathsf{K}^+(\mathcal{A})$ is a triangulated subcategory. Let now $L \in \mathsf{K}^+(\mathcal{L})$ be an acyclic complex. Thus, L decomposes into short exact sequences $0 \to \operatorname{Ker}(d_L^n) \to L^n \xrightarrow{d_L^n} \operatorname{Ker}(d_L^{n+1}) \to 0$ for $n \in \mathbb{Z}$. As L is bounded below, repeated application of (b) shows that each $\operatorname{Ker}(d_L^n)$ lies in $\mathcal{L}$, and hence by (c), the sequences $0 \to F(\operatorname{Ker}(d_L^n)) \to F(L^n) \to F(\operatorname{Ker}(d_L^{n+1})) \to 0$ are exact for all $n \in \mathbb{Z}$. It follows that $F(L)$ is acyclic. Now the assertion follows from Theorem 15.1. $\square$

Remark 15.3. In the context of Theorem 15.2, it follows directly from the computation of $\mathrm{R}F$ in (ii) that $\mathrm{R}^i F = 0$ for all $i < 0$ and that the objects $L \in \mathcal{L}$ are *F-acyclic*, that is, $\mathrm{R}^i F(L) = 0$ for all $i > 0$. Consequently, the δ-functor $(\{\mathrm{R}^i F\}_i, \{\delta^i\}_i)$ is effaceable and hence universal.

Example 15.4. Let $F \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and suppose that $\mathcal{A}$ has enough injectives. Then every short exact sequence $0 \to I \to A \to B \to 0$ in $\mathcal{A}$ with I injective splits; taking $\mathcal{L}$ to be the full subcategory of injective objects, it follows easily that the hypotheses of Theorem 15.2 are satisfied. Hence, the derived functor $\mathrm{R}F \colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists and is computed by $\mathrm{R}F(X) = F(I^\bullet)$ for every $X \in \mathcal{A}$, where $X \to I^\bullet$ is an injective resolution. This recovers the classical definition of derived functors.

Exercise 15.5. Let X be a topological space and consider the global sections functor

$$\Gamma(X, \_) \colon \operatorname{Shv}(X, \mathsf{Ab}) \to \mathsf{Ab}.$$

Let $\mathcal{L} \subseteq \mathrm{Shv}(X, \mathsf{Ab})$ be the full subcategory of *flabby sheaves*, that is, sheaves $\mathcal{F}$ such that for all open subsets $U \subseteq V \subseteq X$ the restriction map $\mathcal{F}(V) \twoheadrightarrow \mathcal{F}(U)$ is surjective.

Show that $\mathcal{L}$ satisfies the hypotheses of Theorem 15.2 and deduce that the right derived functor $\mathrm{R}\Gamma(X, \_)\colon \mathsf{D}^+(\mathrm{Shv}(X, \mathsf{Ab})) \to \mathsf{D}(\mathsf{Ab})$ exists. The higher derived functor $\mathrm{H}^n(X, \_) := \mathrm{H}^n\mathrm{R}\Gamma(X, \_)$ is the usual n-th sheaf cohomology.

In the case of enough injectives, Theorem 15.2 allows the following simplification:

Theorem 15.6. *Let $\mathcal{A}$ be an abelian category with enough injectives, and denote $\mathcal{I} \subseteq \mathcal{A}$ the full subcategory of injective objects. Then*

(i) The functor $Q\colon \mathsf{K}^+(\mathcal{A}) \to \mathsf{D}^+(\mathcal{A})$ is a Bousfield localization. More precisely, the restriction $Q\big|_{\mathsf{K}^+(\mathcal{I})}\colon \mathsf{K}^+(\mathcal{I}) \xrightarrow{\sim} \mathsf{D}^+(\mathcal{A})$ is an equivalence of triangulated categories. We denote by

$$\mathbf{i}\colon \mathsf{D}^+(\mathcal{A}) \xrightarrow{\sim} \mathsf{K}^+(\mathcal{I}) \subseteq \mathsf{K}^+(\mathcal{A})$$

a right adjoint of Q which factors through $\mathsf{K}^+(\mathcal{I})$.

(ii) The unit $\eta\colon \mathrm{id}_{\mathsf{K}^+(\mathcal{A})} \to \mathbf{i}Q$ gives functorial injective resolutions: For every $X \in \mathsf{K}^+(\mathcal{A})$ the map $\eta_X\colon X \to \mathbf{i}Q(X)$ is a quasi-isomorphism into a complex of injectives.

(iii) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Then the right derived functor $\mathrm{R}F$ is computed as the composition

$$\mathsf{D}^+(\mathcal{A}) \xrightarrow{\mathbf{i}} \mathsf{K}^+(\mathcal{I}) \xrightarrow{\mathsf{K}F|_{\mathsf{K}^+(\mathcal{I})}} \mathsf{K}(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}(\mathcal{B}).$$

Proof. Since $\mathcal{A}$ has enough injectives and $\mathsf{K}^+(\mathcal{I})$ clearly satisfies the hypotheses of Theorem 15.2, we deduce that for every $X \in \mathsf{K}^+(\mathcal{A})$ there exists a quasi-isomorphism $X \to I$ with $I \in \mathsf{K}^+(\mathcal{I})$. We need to show that every object $I \in \mathsf{K}^+(\mathcal{I})$ is local with respect to quasi-isomorphisms. Then (i) follows from Proposition 13.9. Alternatively, we can apply Corollary 10.9 to the subcategory $\mathcal{D} = \mathsf{K}^+(\mathcal{I})$. Since $\mathrm{qis} \cap \mathsf{K}^+(\mathcal{I})$ consists of isomorphisms, we deduce that $\mathsf{K}^+(\mathcal{I}) = \mathsf{K}^+(\mathcal{I})_{\mathrm{qis}} \xrightarrow{\sim} \mathsf{D}^+(\mathcal{A})$ is an equivalence of categories. Part (iii) follows from (the proof of) Theorem 15.1 or Proposition 13.10.

Since $\mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(-, I)$ is a cohomological functor by Proposition 4.7, we see that $I \in \mathsf{K}^+(\mathcal{I})$ is local if and only if

$$\mathrm{Hom}_{\mathsf{K}^+(\mathcal{A})}(X, I) = 0$$

for every *acyclic* complex $X \in \mathsf{K}^+(\mathcal{A})$. We even prove this in the case where X is unbounded. So let $f\colon X \to I$ be a map of complexes, where X is acyclic. We need to show that f is null homotopic, which is a standard exercise. Fix i_0 such that $I^i = 0$ for all $i < i_0$. We construct a homotopy $\{s^i\colon X^i \to I^{i-1}\}_i$ inductively as follows: For $i \le i_0$ we put $s^i := 0$. Suppose that for some n we have constructed maps $\{s^i\}_{i \le n}$ such that

$$f^{i-1} = s^i d_X^{i-1} + d_I^{i-2} s^{i-1} \tag{15.1}$$

for all $i \le n$,

$$\begin{tikzcd}
\cdots \arrow[r] & X^{n-2} \arrow[r, "d_X^{n-2}"] \arrow[d, "f^{n-2}"] \arrow[dl] & X^{n-1} \arrow[r, "d_X^{n-1}"] \arrow[d, "f^{n-1}"] \arrow[dl, "s^{n-1}"'] & X^n \arrow[r, "d_X^n"] \arrow[d, "f^n"] \arrow[dl, "s^n"'] & X^{n+1} \arrow[r] \arrow[d, "f^{n+1}"] \arrow[dl, dashed, "s^{n+1}"'] & \cdots \\
\cdots \arrow[r] & I^{n-2} \arrow[r, "d_I^{n-2}"'] & I^{n-1} \arrow[r, "d_I^{n-1}"'] & I^n \arrow[r, "d_I^n"'] & I^{n+1} \arrow[r] & \cdots .
\end{tikzcd}$$

Now, we compute

$$\begin{aligned}(f^n - d_I^{n-1}s^n)\circ d_X^{n-1} &= f^n d_X^{n-1} - d_I^{n-1}s^n d_X^{n-1} = d_I^{n-1}f^{n-1} - d_I^{n-1}s^n d_X^{n-1}\\ &= d_I^{n-1}\circ(f^{n-1} - s^n d_X^{n-1}) = d_I^{n-1}\circ d_I^{n-2}s^{n-1} = 0. \qquad \text{(by (15.1))}.\end{aligned}$$

As X is exact at X^n, we deduce that $f^n - d_I^{n-1}s^n$ factors as $X^n \xrightarrow{d_X^n} \operatorname{Im}(d_X^n) \xrightarrow{\overline{s}} I^n$. As I^n is injective, we can factor $\overline{s}$ as a composite $\operatorname{Im}(d_X^n) \hookrightarrow X^{n+1} \xrightarrow{s^{n+1}} I^{n+1}$. By construction we have $f^n = s^{n+1}d_X^n + d_I^{n-1}s^n$, which finishes the induction step. Therefore, f is null homotopic. □

Remark 15.7. Let $\mathcal{A}$ be an abelian category and $I \in \mathsf{K}^+(\mathcal{A})$ be a complex of injectives and $X \in \mathsf{K}(\mathcal{A})$. The proof of Theorem 15.6(i) shows that I is qis-local and then Proposition 13.9(i) shows that the map

$$\operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) \xrightarrow{\sim} \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(QX, QI)$$

is an isomorphism.

Up until now we have only proved the existence of derived functors in the bounded case. In order to obtain a criterion for unbounded derived functors, we either need to put stronger hypotheses on the functor or on the category. We will explore the first case now and postpone the latter case to Theorem 17.11.

Theorem 15.8. *Let $\mathcal{A}$ be an abelian category and let $\mathcal{L} \subseteq \mathcal{A}$ be a full subcategory satisfying conditions (a) and (b) of Theorem 15.2.*

(i) Suppose that there exists an integer $d \geq 0$ such that for all exact sequences

$$L_0 \to L_1 \to \cdots \to L_{d-1} \to L_d \to 0$$

we have that $L_0, \ldots, L_{d-1} \in \mathcal{L}$ implies $L_d \in \mathcal{L}$.

Then every complex $X \in \mathsf{K}(\mathcal{A})$ admits a quasi-isomorphism $X \to L$ into a complex $L \in \mathsf{K}(\mathcal{L})$.

(ii) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor. Suppose that F satisfies condition (c) in Theorem 15.2(ii) and has finite cohomological dimension, *i.e., there exists an integer $d \geq 0$ such that $\mathrm{R}^i F = 0$ for all $i > d$.*

Then the right derived functor $\mathrm{R}F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists and can be computed as

$$\mathrm{R}F(Q_{\mathcal{A}}(X)) = Q_{\mathcal{B}}\mathsf{K}F(L),$$

where $X \to L$ is any quasi-isomorphism with $L \in \mathsf{K}(\mathcal{L})$.

Proof. We first show how to deduce (ii) from (i). By Theorem 15.2 the bounded derived functor $\mathrm{R}F\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ exists. Hence also the higher derived functors $\mathrm{R}^i F$ exist. We now apply (i) to the full subcategory $\mathcal{L}' \subseteq \mathcal{A}$ consisting of F-acyclic objects. We need to check that conditions (a), (b) and (c) of Theorem 15.2 are satisfied. Since $\mathcal{L} \subseteq \mathcal{L}'$ by Remark 15.3, it is clear that every object $A \in \mathcal{A}$ admits a monomorphism $A \hookrightarrow L$ with $L \in \mathcal{L}'$, whence (a). Let $0 \to L \to A \to B \to 0$ be a short exact sequence in $\mathcal{A}$ such that L is F-acyclic. The long exact sequence in cohomology then gives a short exact sequence $0 \to F(L) \to F(A) \to F(B) \to \mathrm{R}^1F(L) = 0$ and isomorphisms

$\mathrm{R}^i F(A) \xrightarrow{\sim} \mathrm{R}^i F(B)$ for all $i \geq 1$. Hence, (b) and (c) are satisfied. Let now $L_0 \to L_1 \to \cdots \to L_{d-1} \to L_d \to 0$ be an exact sequence such that $L_0, \dots, L_{d-1}$ are F-acyclic. We split the sequence into short exact sequences $0 \to M_i \to L_i \to M_{i+1} \to 0$ for $0 \leq i < d$, where $M_d = L_d$. Since each L_i is F-acyclic, the long exact sequence in cohomology gives isomorphisms

$$\mathrm{R}^i F(L_d) \cong \mathrm{R}^{i+1} F(M_{d-1}) \cong \cdots \cong \mathrm{R}^{i+d} F(M_0) = 0$$

for all $i > 0$. Hence, L_d is F-acyclic. Together with (c) we deduce that, if $L \in \mathsf{K}(\mathcal{L}')$ is acyclic, then so is $\mathsf{K}F(L)$. By (i), every $X \in \mathsf{K}(\mathcal{A})$ admits a quasi-isomorphism $X \to L$ with $L \in \mathsf{K}(\mathcal{L}')$. Now, (ii) follows from Theorem 15.1.

It remains to prove (i). We proceed in several steps.

Step 1: For any $X \in \mathsf{K}(\mathcal{A})$ and $n \in \mathbb{Z}$ there exists a quasi-isomorphism $X \to Y$ such that $Y^i \in \mathcal{L}$ for all $i \geq n$.

Indeed, consider the brutal truncation

$$\sigma^{\geq n} X \coloneqq [\cdots 0 \to 0 \to X^n \to X^{n+1} \to \cdots].$$

Now, Theorem 15.2(i) provides a quasi-isomorphism $\sigma^{\geq n} X \to Z$ with $Z \in \mathsf{K}^+(\mathcal{L})$. Composing with the quasi-isomorphism $Z \to \tau^{\geq n} Z$, and noting that $(\tau^{\geq n} Z)^n = \operatorname{Coker}(d_Z^{n-1}) \in \mathcal{L}$ (this follows from the assumption in (i)), we may assume $Z^i = 0$ for all $i < n$. Now, the spliced complex

$$Y \coloneqq [\cdots X^{n-2} \to X^{n-1} \xrightarrow{d^{n-1}} Z^n \to Z^{n+1} \to \cdots],$$

where d^{n-1} is given as the composition $X^{n-1} \to X^n \to Z^n$, is as desired.

Step 2: Fix integers $m < n$ and let $X \in \mathsf{K}(\mathcal{A})$ such that $X^i \in \mathcal{L}$ for all $i \geq n$. Then there exists a quasi-isomorphism $X \to Y$ such that $Y^i \in \mathcal{L}$ for all $i \geq m$ and $X^j \xrightarrow{\sim} Y^j$ is an isomorphism for all $j \geq n + 1 + d$.

By Step 1 we find a quasi-isomorphism $f\colon X \to Z$ such that $Z^i \in \mathcal{L}$ for all $i \geq m$. By Lemma 12.2(ii) the mapping cone $M \coloneqq \mathrm{Mc}(f)$ is acyclic. In other words, $\operatorname{Coker}(d_M^{i-2}) \xrightarrow{\sim} \operatorname{Ker}(d_M^i)$ is an isomorphism for all i. Note that $M^i = X^{i+1} \oplus Z^i \in \mathcal{L}$ for all $i \geq n-1$. Moreover, from the exact sequence

$$M^i \to M^{i+1} \cdots \to M^{i+d-1} \to \operatorname{Ker}(d_M^{i+d}) \to 0$$

and our assumption, we deduce $\operatorname{Ker}(d_M^{i+d}) \in \mathcal{L}$ for all $i \geq n-1$. Moreover, contemplating the diagram

$$\begin{tikzcd}
M^i \arrow[d, "d_M^i"'] & = & X^{i+1} \arrow[d, "-d_X^{i+1}"'] \arrow[dr, "f^{i+1}"] & \oplus & Z^i \arrow[d, "d_Z^i"] \\
M^{i+1} & = & X^{i+2} & \oplus & Z^{i+1}
\end{tikzcd}$$

we observe that $\operatorname{Ker}(d_M^i) = \operatorname{Ker}(d_X^{i+1}) \times_{Z^{i+1}} Z^i$ and $\operatorname{Coker}(d_M^i) = X^{i+2} \sqcup_{X^{i+1}} \operatorname{Coker}(d_Z^i)$.

We put $a \coloneqq n + d$. To summarize, we have an isomorphism

$$\operatorname{Coker}(d_M^{a-2}) = X^a \sqcup_{X^{a-1}} \operatorname{Coker}(d_Z^{a-2}) \xrightarrow{\sim} \operatorname{Ker}(d_X^{a+1}) \times_{Z^{a+1}} Z^a = \operatorname{Ker}(d_M^a)$$

in $\mathcal{L}$ and a commutative diagram

$$\begin{tikzcd}
X^{a-1} \arrow[r] \arrow[d] & X^a \arrow[r] \arrow[d] & X^{a+1} \arrow[dd, equal] \\
Z^{a-1} \arrow[r] \arrow[dd, equal] & \mathrm{Coker}(d_M^{a-2}) \arrow[d, "\sim"] & \\
 & \mathrm{Ker}(d_M^a) \arrow[r] \arrow[d] & X^{a+1} \arrow[d] \\
Z^{a-1} \arrow[r] & Z^a \arrow[r] & Z^{a+1}.
\end{tikzcd}$$

Hence, defining Y to be the complex

$$Y \coloneqq [\cdots \to Z^{a-2} \to Z^{a-1} \xrightarrow{d_Y^{a-1}} \mathrm{Ker}(d_M^a) \xrightarrow{d_Y^a} X^{a+1} \to X^{a+2} \to \cdots],$$

we have maps $X \to Y \to Z$, $Y^i \in \mathcal{L}$ for all $i \geq m$ and $X^i \xrightarrow{\sim} Y^i$ is an isomorphism for all $i \geq a+1 = n+d+1$. It remais to show that $g \colon X \to Y$ is a quasi-isomorphism. It is obvious that $\mathrm{H}^i(g)$ is an isomorphism whenever $i \notin \{a-1, a, a+1\}$, so it remains to treat these three cases.

Note that $\mathrm{Im}(d_Y^a) = \mathrm{Im}\big(X^a \sqcup_{X^{a-1}} \mathrm{Coker}(d_Z^{a-2}) \xrightarrow{(d_X^a, 0)} X^{a+1}\big) = \mathrm{Im}(d_X^a)$, from which we deduce $\mathrm{H}^{a+1}(Y) \cong \mathrm{H}^{a+1}(X)$. Similarly, we have $\mathrm{Ker}(d_Y^{a-1}) = \mathrm{Ker}\big(Z^{a-1} \xrightarrow{(0, d_Z^{a-1})} \mathrm{Ker}(d_X^a) \times_{Z^{a+1}} Z^a\big) = \mathrm{Ker}(d_Z^{a-1})$, and hence $\mathrm{H}^{a-1}(Y) \cong \mathrm{H}^{a-1}(Z) \cong \mathrm{H}^{a-1}(X)$. Finally, we show that $\mathrm{H}^a(X \to Y)$ is an isomorphism. First note that

$$\begin{tikzcd}
 & X^{a-1} \arrow[r, "d_X^{a-1}"] \arrow[d] & X^a \arrow[d] \\
Z^{a-1} \arrow[r, two heads] \arrow[rr, hook, bend right, "d_Y^{a-1}"'] & \mathrm{Coker}(d_Z^{a-2}) \arrow[r] & Y^a \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

is a pushout diagram, and hence $\mathrm{Coker}(d_X^{a-1}) \xrightarrow{\sim} \mathrm{Coker}(d_Y^{a-1})$ by Proposition 3.10(ii). Now we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{H}^a(X) \arrow[r] \arrow[d] & \mathrm{Coker}(d_X^{a-1}) \arrow[r] \arrow[d, "\sim"] & X^{a+1} \arrow[d, "\mathrm{id}"] \\
0 \arrow[r] & \mathrm{H}^a(Y) \arrow[r] & \mathrm{Coker}(d_Y^{a-1}) \arrow[r] & Y^{a+1},
\end{tikzcd}$$

where the rows are exact. The five lemma shows that $\mathrm{H}^a(X \to Y)$ is an isomorphism.

Step 3: Completion of the proof. Let $X \in \mathsf{K}(\mathcal{A})$. Take an infinite sequence of integers $n_0 > n_1 > n_2 > \cdots$. Step 1 provides a quasi-isomorphism $X \to Y_0$ such that $Y_0^i \in \mathcal{L}$ for all $i \geq n_0$. By Step 2, we inductively construct a sequence $Y_0 \to Y_1 \to Y_2 \to \cdots$ of quasi-isomorphisms such that $Y_k^i \in \mathcal{L}$ for all $i \geq n_k$ and such that $Y_k^j \xrightarrow{\sim} Y_{k+1}^j$ is an isomorphism for all $j \geq n_k+1+d$. We now consider the complex $Y \coloneqq \varinjlim_k Y_k$ in $\mathsf{C}(\mathcal{A})$. Then $Y \in \mathsf{K}(\mathcal{L})$ and $X \to Y$ is a quasi-isomorphism: Indeed, for any $n \in \mathbb{Z}$, pick k such that $n_k + 1 + d < n$, so that $Y^i = Y_k^i \in \mathcal{L}$ for all $i \geq n-1$. Then $\mathrm{H}^n(Y_k \to Y)$ is an isomorphism, hence so is $\mathrm{H}^n(X \to Y)$. $\square$

§16. The Ext-functor

We apply the machinery in the last section to determine the right derived functors of the Hom functor.

Definition 16.1. Let $\mathcal{A}$ be an abelian category with enough injectives, and let $A \in \mathcal{A}$. The higher right derived functor $\mathrm{Ext}^n_{\mathcal{A}}(A, \_) \coloneqq \mathrm{R}^n \operatorname{Hom}_{\mathcal{A}}(A, \_)\colon \mathcal{A} \to \mathsf{Ab}$ is called the *n-th Ext* functor.

Note the functor $\mathrm{Ext}^n_{\mathcal{A}}(A, \_)$ is natural in A (*e.g.*, by Remark 13.2), so that we obtain a bifunctor

$$\mathrm{Ext}^n_{\mathcal{A}}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \longrightarrow \mathsf{Ab}.$$

In practice it turns out that the derived functors of many functors (namely, the *representable* ones) can be described in terms of the Ext-functor:

Example 16.2. Let X be a topological space. Denoting by $\underline{\mathbb{Z}} \in \mathrm{Shv}(X, \mathsf{Ab})$ the constant sheaf $\mathbb{Z}$, we have a natural isomorphism $\Gamma(X, \mathcal{F}) \cong \operatorname{Hom}(\underline{\mathbb{Z}}, \mathcal{F})$ for every sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathsf{Ab})$. Thus, for each $n \geq 0$ we deduce a natural isomorphism

$$\mathrm{Ext}^n_{\mathrm{Shv}(X,\mathsf{Ab})}(\underline{\mathbb{Z}}, \mathcal{F}) \cong \mathrm{H}^n(X, \mathcal{F}).$$

Exercise 16.3. Let $\mathcal{A}$ be an abelian category and fix $A, B \in \mathcal{A}$. An *extension of A by B* is defined to be a short exact sequence $(f, E, g) \coloneqq [0 \to B \xrightarrow{f} E \xrightarrow{g} A \to 0]$. Given two extensions (f, E, g) and (f', E', g'), we write $(f, E, g) \sim (f', E', g')$ if there exists a map $h\colon E \to E'$ such that $f' = h \circ f$ and $g' \circ h = g$.

(i) Show that $\sim$ is an equivalence relation.

(ii) Let $\mathrm{Ext}(A, B)$ be the set of equivalence classes of extensions of A by B. Show that the assignment $(A, B) \mapsto \mathrm{Ext}(A, B)$ enhances to a bifunctor $\mathrm{Ext}\colon \mathcal{A}^{\mathrm{op}} \times \mathcal{A} \to \mathsf{Set}$.

(iii) Show that $\mathrm{Ext}(A, B)$ is canonically an abelian group.

(iv) Suppose that $\mathcal{A}$ has enough injectives. Construct a natural isomorphism

$$\mathrm{Ext}(A, B) \cong \mathrm{Ext}^1_{\mathcal{A}}(A, B)$$

of abelian groups.

Our goal in this section is to show that Ext-groups have a natural description inside the derived category of an abelian category. To this end, we make the following general definition:

Definition 16.4. Let $\mathcal{A}$ be an abelian category, and let $X, Y \in \mathsf{D}(\mathcal{A})$. For any $n \in \mathbb{Z}$, the abelian group

$$\mathbb{E}\mathrm{xt}^n(X, Y) \coloneqq \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(X, Y[n])$$

is called the *n-th Hyperext* of X, Y.

Definition 16.5. Let $\mathcal{A}$ be an abelian category and fix $X, Y \in \mathsf{K}(\mathcal{A})$. We define a complex $\operatorname{Hom}^\bullet(X, Y)$ via

$$\operatorname{Hom}^n(X, Y) \coloneqq \prod_{i \in \mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Y^{i+n})$$

for all $n \in \mathbb{Z}$. The differential $d^n\colon \operatorname{Hom}^n(X,Y) \to \operatorname{Hom}^{n+1}(X,Y)$ is defined by

$$(d^n f)_i := d_Y^{n+i} f_i - (-1)^n f_{i+1} d_X^i$$

as a map $X^i \to Y^{i+n+1}$.

Lemma 16.6. *Let $\mathcal{A}$ be an abelian category and fix $X \in \mathsf{K}(\mathcal{A})$. Then:*

(i) The assignment $Y \mapsto \operatorname{Hom}^\bullet(X,Y)$ induces an exact functor $\operatorname{Hom}^\bullet(X,-)\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}(\mathsf{Ab})$.

(ii) For every $Y \in \mathsf{K}(\mathcal{A})$ and $n \in \mathbb{Z}$ we have

$$\mathrm{H}^n(\operatorname{Hom}^\bullet(X,Y)) = \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X, Y[n]).$$

(iii) Suppose that $\mathcal{A}$ has enough injectives. Let $Y \in \mathsf{K}^+(\mathcal{A})$ be a complex of injective objects and suppose one of Y, X is acyclic. Then $\operatorname{Hom}^\bullet(X,Y)$ is acyclic.

Proof. We first prove (i). It is clear that $\operatorname{Hom}^\bullet(X,-)\colon \mathsf{C}(\mathcal{A}) \to \mathsf{C}(\mathsf{Ab})$ is a functor. Let $\varphi\colon Y \to Z$ be a null homotopic map and fix a null homotopy s, so that $\varphi = d_Z s + s d_Y$. We claim that the induced maps $s_*^n\colon \operatorname{Hom}^n(X,Y) \to \operatorname{Hom}^{n-1}(X,Z)$ define a null homotopy of $\varphi_* = \operatorname{Hom}^\bullet(X,\varphi)$. We compute

$$\begin{aligned}\varphi_*(f) = \varphi f = d_Z s f + s d_Y f &= d_Z s f + s d^n(f) + (-1)^n s f d_X = d_Z s f - (-1)^{n-1} s f d_X + s d^n(f)\\ &= d^{n-1}(sf) + s d^n(f) = (d^{n-1} \circ s_* + s_* \circ d^n)(f),\end{aligned}$$

for any $f \in \operatorname{Hom}^n(X,Y)$, hence φ_* is indeed null homotopic. We conclude that $\operatorname{Hom}^\bullet(X,-)$ descends to a functor on the homotopy categories.

We now address the exactness. Define a natural isomorphism

$$\begin{aligned}\xi\colon \operatorname{Hom}^n(X,Y[1]) = \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Y^{n+i+1}) &\xrightarrow{\sim} \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Y^{n+1+i}) = \operatorname{Hom}^n(X,Y)[1]\\ (f_i)_i &\longmapsto (f_i)_i;\end{aligned}$$

it is trivial to check that ξ preserves the differentials. Let now $g\colon Y \to Z$ be a map of complexes. We construct an isomorphism

$$\alpha\colon \operatorname{Hom}^\bullet(X, \operatorname{Mc}(g)) \xrightarrow{\sim} \operatorname{Mc}\big(\operatorname{Hom}^\bullet(X,g)\big)$$

via

$$\begin{aligned}\prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Y^{n+i+1} \oplus Z^{n+i}) &\xrightarrow{\sim} \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Y^{n+1+i}) \oplus \prod_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{A}}(X^i, Z^{n+i}),\\ (f_i)_i &\longmapsto \big((f_i^Y)_i, (f_i^Z)_i\big),\end{aligned}$$

where f_i^Y denotes the composition of f_i with the projection to Y^{n+1+i}, and likewise for f_i^Z. We need to check that this map preserves the differentials. To this end, let $f \in \operatorname{Hom}^n(X, \operatorname{Mc}(g))$. On the one hand we compute

$$\begin{aligned}(d^n f)_i &= d^{n+i}_{\operatorname{Mc}(g)} \circ f_i - (-1)^n f_{i+1} \circ d_X^i\\ &= \begin{pmatrix} -d_Y^{n+i+1} & 0 \\ g^{n+i} & d_Z^{n+i} \end{pmatrix} \circ \begin{pmatrix} f_i^Y \\ f_i^Z \end{pmatrix} - (-1)^n \begin{pmatrix} f_{i+1}^Y \\ f_{i+1}^Z \end{pmatrix} \circ d_X^i\\ &= \begin{pmatrix} -d_Y^{n+i+1} \circ f_i^Y - (-1)^n f_{i+1}^Y \circ d_X^i \\ g^{n+i} \circ f_i^Y + d_Z^{n+i} \circ f_i^Z - (-1)^n f_{i+1}^Z \circ d_X^i \end{pmatrix}.\end{aligned}$$

On the other hand, we compute

$$\left[\begin{pmatrix} d^n_{\mathrm{Hom}(X,Y)[1]} & 0 \\ g & d^n_{\mathrm{Hom}(X,Z)} \end{pmatrix}\begin{pmatrix} f^Y \\ f^Z \end{pmatrix}\right]_i = \begin{pmatrix} -d^{n+1}_{\mathrm{Hom}(X,Y)}(f^Y) \\ g \circ f^Y + d^n_{\mathrm{Hom}(X,Z)}(f^Z) \end{pmatrix}_i = \begin{pmatrix} (-1)\cdot(d_Y^{n+1+i} \circ f_i^Y - (-1)^{n+1} f^Y_{i+1} \circ d^i_X) \\ g^{n+i} \circ f_i^Y + d_Z^{n+i} \circ f_i^Z - (-1)^n f^Z_{i+1} \circ d^i_X \end{pmatrix}$$

and observe that both expressions agree. Hence α is indeed an isomorphism of complexes.

Now, given the distinguished triangle $Y \xrightarrow{u} Z \xrightarrow{v} \mathrm{Mc}(u) \xrightarrow{w} Y[1]$ in $\mathsf{K}(\mathcal{A})$, we obtain an isomorphism of triangles

$$\begin{tikzcd}
\mathrm{Hom}^\bullet(X,Y) \arrow[r, "u_*"] \arrow[d, equal] & \mathrm{Hom}^\bullet(X,Z) \arrow[r, "v_*"] \arrow[d, equal] & \mathrm{Hom}^\bullet(X,\mathrm{Mc}(u)) \arrow[r, "w_*"] \arrow[d, "\sim"', "\alpha"] & \mathrm{Hom}^\bullet(X,Y[1]) \arrow[d, "\sim"', "\xi"] \\
\mathrm{Hom}_\bullet(X,Y) \arrow[r, "u_*"'] & \mathrm{Hom}^\bullet(X,Z) \arrow[r] & \mathrm{Mc}\big(\mathrm{Hom}^\bullet(X,u)\big) \arrow[r] & \mathrm{Hom}^\bullet(X,Y)[1],
\end{tikzcd}$$

where we note that the diagram already commutes in $\mathsf{C}(\mathcal{A})$. This finishes the proof that $\mathrm{Hom}^\bullet(X,\_)$ is exact.

Let us now prove (ii). Since $\mathrm{H}^n(\mathrm{Hom}^\bullet(X,Y)) = \mathrm{H}^0(\mathrm{Hom}^\bullet(X,Y)[n]) = \mathrm{H}^0(\mathrm{Hom}^\bullet(X,Y[n]))$, we reduce to the case $n=0$. By definition we have

$$\mathrm{Ker}\big(d^0\colon \mathrm{Hom}^0(X,Y) \to \mathrm{Hom}^1(X,Y)\big) = \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X,Y)$$

For any $s \in \mathrm{Hom}^{-1}(X,Y)$ we compute $(d^{-1}s)_i = d_Y^{i-1}s_i + s_{i+1}d^i_X$, hence we conclude that the image of $d^{-1}\colon \mathrm{Hom}^{-1}(X,Y) \to \mathrm{Hom}^0(X,Y)$ consists of the null homotopic maps. We conclude that $\mathrm{H}^0(\mathrm{Hom}^\bullet(X,Y)) = \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y)$.

For part (iii), let $Y \in \mathsf{K}^+(\mathcal{A})$ be a complex of injective objects and suppose that either X or Y is acyclic. In view of (ii), it suffices to show $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y[n]) = 0$. But by Remark 15.7 we have

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,Y[n]) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(QX,QY[n]) = 0.$$

□

Theorem 16.7. *Let $\mathcal{A}$ be an abelian category with enough injectives.*

(i) For every $X \in \mathsf{K}(\mathcal{A})$ the derived functor

$$\mathrm{R}\,\mathrm{Hom}^\bullet(X,\_)\colon \mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathsf{Ab})$$

exists.

(ii) For every $X \in \mathsf{K}(\mathcal{A})$, $Y \in \mathsf{D}^+(\mathcal{A})$ and $n \in \mathbb{Z}$ we have

$$\mathrm{H}^n\mathrm{R}\,\mathrm{Hom}^\bullet(X,Y) = \mathrm{Hom}_{\mathsf{D}(\mathcal{A})}(QX,Y[n]) = \mathbb{E}\mathrm{xt}^n(QX,Y).$$

(iii) For every $X, Y \in \mathcal{A}$ and $n \in \mathbb{Z}$, we have

$$\mathrm{Ext}^n_{\mathcal{A}}(X,Y) = \mathbb{E}\mathrm{xt}^n(X,Y).$$

Proof. For (i), apply Theorem 15.1 to the full subcategory $\mathcal{L} \subseteq \mathsf{K}^+(\mathcal{A})$ of complexes of injectives, noting that the hypotheses are satisfied by Theorem 15.6(i) and Lemma 16.6(iii), respectively.

For (ii), let $X \in \mathsf{K}(\mathcal{A})$, $Y \in \mathsf{D}^+(\mathcal{A})$, and $n \in \mathbb{Z}$. Then

$$\mathrm{H}^n \mathrm{R}\operatorname{Hom}^\bullet(X, Y) = \mathrm{H}^n \operatorname{Hom}^\bullet(X, \mathbf{i}Y) = \operatorname{Hom}_{\mathsf{K}(\mathcal{A})}(X, \mathbf{i}Y[n]) = \operatorname{Hom}_{\mathsf{D}(\mathcal{A})}(QX, Y[n]).$$

Finally, for (iii), we note that for $X \in \mathcal{A}$, we have $\operatorname{Hom}^\bullet(X, \_) = \mathsf{K}\operatorname{Hom}_{\mathcal{A}}(X, \_)$ as functors $\mathsf{K}^+(\mathcal{A}) \to \mathsf{K}(\mathsf{Ab})$. Hence, the claim follows from (ii). $\square$

§17. Homotopy limits and homotopy colimits

In general, a triangulated category need not admit arbitrary (co)limits. However, it turns out that, as soon as countable (co)products exist, one can define the related notion of homotopy (co)limit. In practice these are quite well behaved although they do not satisfy a universal property.

Lemma 17.1. *Let $\mathcal{C}$ be a triangulated category. Suppose that $\mathcal{C}$ has arbitrary direct sums (resp. products). Then arbitrary direct sums (resp. products) of distinguished triangles are distinguished.*

Proof. This is almost literally the same proof as for Proposition 5.2. $\square$

Lemma 17.2. *Let $\mathcal{C}$ be a triangulated category and let $\mathcal{N} \subseteq \mathcal{C}$ be a triangulated subcategory.*

(i) Suppose that $\mathcal{C}$ has arbitrary direct sums. If $\mathcal{N}$ is closed under all direct sums then the localization functor $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ preserves these. In particular, also $\mathcal{C}/\mathcal{N}$ has direct sums.

Conversely, if $\mathcal{N}$ is thick and Q preserves arbitrary direct sums, then $\mathcal{N}$ is closed under these.

(ii) Suppose that $\mathcal{C}$ has arbitrary products. If $\mathcal{N}$ is closed under all products, then the localization functor $Q\colon \mathcal{C} \to \mathcal{C}/\mathcal{N}$ preserves these. In particular, also $\mathcal{C}/\mathcal{N}$ has products.

Conversely, if $\mathcal{N}$ is thick and Q preserves arbitrary products, then $\mathcal{N}$ is closed under these.

Proof. Note that (ii) is dual to (i). For the "only if"-direction consider the set $S_{\mathcal{N}}$ of morphisms $X \to Y$ whose cone lies in $\mathcal{N}$ contains all identities and satisfies the condition

($*$) If $(s_i)_{i\in I}$ is a collection of morphisms in $S_{\mathcal{N}}$, then $\bigoplus_{i\in I} s_i \in S_{\mathcal{N}}$

by Lemma 17.1. Now the same proof as for Proposition 8.10 applies *mutatis mutandis.* (Observe that Lemma 8.7 also holds for infinite products of categories.)

The converse follows from the observation that $\mathcal{N}$ identifies with $\operatorname{Ker}(Q)$ by Remark 11.4, which is closed under arbitrary direct sums because Q commutes with these. $\square$

Example 17.3. Let $\mathcal{A}$ be an abelian category.

(a) If $\mathcal{A}$ is AB3 (infinite direct sums exist), then $\mathsf{K}(\mathcal{A})$ has direct sums (which are given by direct sums of complexes).

(b) If $\mathcal{A}$ is AB4 (infinite direct sums exist and are exact), then $\mathsf{D}(\mathcal{A})$ has direct sums (which are given by direct sums of complexes).

Definition 17.4. Let $(\mathcal{C}, T)$ be a triangulated category.

(a) If $\mathcal{C}$ has countable direct sums, then the *homotopy colimit* of a sequence

$$X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} \cdots,$$

is the object $\operatorname{hocolim}_i X_i$ in $\mathcal{C}$ defined by the distinguished triangle

$$\bigoplus_{i=0}^{\infty} X_i \xrightarrow{\text{id}-\text{shift}} \bigoplus_{i=0}^{\infty} X_i \longrightarrow \operatorname*{hocolim}_{i\geq 0} X_i \longrightarrow T\Big(\bigoplus_{i=0}^{\infty} X_i\Big);$$

here, the map id−shift is given by $X_n \xrightarrow{(\text{id},-f_n)} X_n \oplus X_{n+1} \hookrightarrow \bigoplus_{i=0}^{\infty} X_i$ on the n-th component. Informally, the map sends $(x_0, x_1, x_2, \dots) \mapsto (x_0, x_1 - f_0(x_0), x_2 - f_1(x_1), \dots)$. Note that the homotopy colimit is unique up to (non-unique) isomorphism.

(b) If $\mathcal{C}$ has countable products, then the *homotopy limit* of a sequence

$$\cdots \xrightarrow{f_3} X_2 \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$$

is the object $\operatorname{holim}_i X_i$ in $\mathcal{C}$ defined by the distinguished triangle

$$\operatorname*{holim}_{i\geq 0} X_i \longrightarrow \prod_{i=0}^{\infty} X_i \xrightarrow{\text{id}-\text{shift}} \prod_{i=0}^{\infty} X_i \longrightarrow T\big(\operatorname*{holim}_{i\geq 0} X_i\big);$$

here, the map id − shift is given by $\prod_{i=0}^{\infty} X_i \twoheadrightarrow X_{n+1} \oplus X_n \xrightarrow{(-f_{n+1},\text{id})} X_n$ on the n-th component. Informally, the map sends $(\dots, x_1, x_0) \mapsto (\dots, x_1 - f_2(x_2), x_0 - f_1(x_1))$. Note that the homotopy limit is unique up to (non-unique) isomorphism.

Remark 17.5. Let $(\mathcal{C}, T)$ be a triangulated category which has countable direct sums. Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$ be a sequence of maps so that the homotopy colimit $\operatorname{hocolim}_i X_i$ is defined.

- By the definition of $\operatorname{hocolim}_i X_i$ we have maps $\iota_n \colon X_n \to \operatorname{hocolim}_i X_i$ such that $\iota_n = \iota_{n+1} \circ f_n$ for all $n \geq 0$.

- Let $M \in \mathcal{C}$ be any object. Since $\operatorname{Hom}(-, M)$ is cohomological (Proposition 4.7), we obtain a long exact sequence

$$\cdots \to \prod_{i=0}^{\infty} \operatorname{Hom}(T^{-1}(X_i), M) \xrightarrow{\partial} \operatorname{Hom}(\operatorname{hocolim}_i X_i, M) \to \prod_{i=0}^{\infty} \operatorname{Hom}(X_i, M) \xrightarrow{(\text{id}-\text{shift})^*} \prod_{i=0}^{\infty} \operatorname{Hom}(X_i, M) \to \cdots.$$

  We deduce that any sequence of maps $\nu_n \colon X_n \to M$, such that $\nu_n = \nu_{n+1} \circ f_n$ for all $n \geq 0$, can be extended to a map $\nu \colon \operatorname{hocolim}_i X_i \to M$ which is unique in $\operatorname{Hom}(\operatorname{hocolim}_i X_i, M)$ modulo the image of ∂.

Example 17.6. Let $(\mathcal{C}, T)$ be a triangulated category with countable direct sums and $X \in \mathcal{C}$. The homotopy colimit of $X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} \cdots$ is X. Similarly, if $\mathcal{C}$ has countable products, then the homotopy limit of $\cdots \xrightarrow{\text{id}} X \xrightarrow{\text{id}} X \xrightarrow{\text{id}} X$ is X.

Proof. We will only prove the statement about the homotopy colimit, because the other one is completely dual. The map

$$X \oplus \bigoplus_{i=0}^{\infty} X \xrightarrow{(\iota_0, \mathrm{id}-\mathrm{shift})} \bigoplus_{i=0}^{\infty} X,$$
$$(x, x_0, x_1, x_2, \dots) \longmapsto (x + x_0, x_1 - x_0, x_2 - x_1, \dots)$$

is an isomorphism: The inverse is given on the n-th component by the composite

$$X \xrightarrow{(\mathrm{id}, -\mathrm{id})} X \oplus X \xrightarrow{\mathrm{id} \oplus \Delta} X \oplus \bigoplus_{i=1}^{n} X \hookrightarrow \bigoplus_{i=0}^{\infty} X$$

More informally, the inverse is given by sending

$$(a_0, a_1, a_2, \dots) \mapsto \Big(a, a_0 - a, a_0 + a_1 - a, \cdots, \sum_{i=0}^{n} a_i - a, \dots\Big),$$

where $a \coloneqq \sum_{i=0}^{\infty} a_i$. We thus obtain a split triangle

$$\bigoplus_{i=0}^{\infty} X \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{\infty} X \xrightarrow{(\mathrm{id})_i} X \xrightarrow{0} T\Big(\bigoplus_{i=0}^{\infty} X\Big),$$

which is distinguished by Proposition 5.3(ii). □

Exercise 17.7. Let $\mathcal{C}$ be a triangulated category.

(i) Let $X \in \mathcal{C}$. Show that the homotopy colimit of $X \xrightarrow{0} X \xrightarrow{0} X \xrightarrow{0} \cdots$ is 0

(ii) Homotopy colimits commute with direct sums: Consider two sequences $(X_i, X_i \xrightarrow{f_i} X_{i+1})_{i \geq 0}$ and $(Y_i, Y_i \xrightarrow{g_i} Y_{i+1})_{i \geq 0}$ in $\mathcal{C}$. Show that there is an isomorphism

$$\operatorname*{hocolim}_i (X_i \oplus Y_i) \xrightarrow{\sim} \operatorname*{hocolim}_i X_i \oplus \operatorname*{hocolim}_i Y_i.$$

(iii) Let $(X_i, X_i \xrightarrow{f_i} X_{i+1})_{i \geq 0}$ be a sequence in $\mathcal{C}$. Show that the induced map $\mathrm{hocolim}_{i \geq a} X_i \xrightarrow{\sim} \mathrm{hocolim}_{i \geq 0} X_i$ is an isomorphism in $\mathcal{C}$ (Hint: apply Lemma 4.13.)

Lemma 17.8. *Let $(\mathcal{C}, T)$ be a triangulated category which admits countable direct sums, $\mathcal{A}$ an abelian category which is AB5 (filtered colimits exist and are exact), and let $H \colon \mathcal{C} \to \mathcal{A}$ be a cohomological functor which preserves countable direct sums.*

Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$ be a sequence in $\mathcal{C}$ and denote by $X = \mathrm{hocolim}_i X_i$ the homotopy colimit. Then the map

$$\operatorname*{colim}_i H(X_i) \xrightarrow{\sim} H(X)$$

is an isomorphism.

Proof. Since H is cohomological and commutes with direct sums, we obtain a long exact sequence

$$\cdots \to \bigoplus_{i=0}^{\infty} H(X_i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{\infty} H(X_i) \to H(X) \to \bigoplus_{i=0}^{\infty} H(T(X_i)) \to \cdots .$$

For each $n \geq 0$ the map $\bigoplus_{i=0}^{n} H(X_i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{n+1} H(X_i)$ is a monomorphism. As $\mathcal{A}$ is AB5, we deduce that $\mathrm{id}-\mathrm{shift}\colon \bigoplus_{i=0}^{\infty} H(X_i) \to \bigoplus_{i=0}^{\infty} H(X_i)$ is a monomorphism. Hence, the sequence

$$0 \to \bigoplus_{i=0}^{\infty} H(X_i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{\infty} H(X_i) \to H(X) \to 0$$

is exact, which proves the assertion. □

Proposition 17.9. *Let $\mathcal{A}$ be an abelian category.*

(i) Suppose that $\mathcal{A}$ is AB4 (infinite direct sums exist and are exact). Let $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$ be a sequence in $\mathsf{D}(\mathcal{A})$, let $X \in \mathsf{D}(\mathcal{A})$, and let

$$f\colon \operatorname*{hocolim}_i X_i \to X$$

be a map. Suppose that for each $n \in \mathbb{Z}$ and $i \gg 0$ the map $\mathrm{H}^n(X_i) \xrightarrow{\sim} \mathrm{H}^n(X)$ is an isomorphism. Then f is an isomorphism in $\mathsf{D}(\mathcal{A})$.

(ii) Suppose that $\mathcal{A}$ is AB4 (infinite products exist and are exact). Let $\cdots \xrightarrow{f_2} X_1 \xrightarrow{f_1} X_0$ be a sequence in $\mathsf{D}(\mathcal{A})$, let $X \in \mathcal{D}(\mathcal{A})$, and let*

$$f\colon X \to \operatorname*{holim}_i X_i$$

be a map. Suppose that for each $n \in \mathbb{Z}$ and $i \gg 0$ the map $\mathrm{H}^n(X) \to \mathrm{H}^n(X_i)$ is an isomorphism. Then f is an isomorphism in $\mathsf{D}(\mathcal{A})$.

Proof. We only prove (i); part (ii) is completely dual. It suffices to show that $\mathrm{H}^n(f)$ is an isomorphism for every n. Fix n. In view of Exercise 17.7(iii) we may assume that $\mathrm{H}^n(X_i) \xrightarrow{\sim} \mathrm{H}^n(X)$ and $\mathrm{H}^{n+1}(X_i) \xrightarrow{\sim} \mathrm{H}^{n+1}(X)$ are isomorphisms for all $i \geq 0$. Observe that H^n is cohomological and commutes with countable direct sums since $\mathcal{A}$ is AB4. In view of Example 17.6 we have a morphism of distinguished triangles

$$\begin{tikzcd}
\bigoplus_{i=0}^{\infty} X_i \arrow[r, "\mathrm{id}-\mathrm{shift}"] \arrow[d] & \bigoplus_{i=0}^{\infty} X_i \arrow[r] \arrow[d] & \operatorname{hocolim}_i X_i \arrow[r] \arrow[d, "f"] & T\Big(\bigoplus_{i=0}^{\infty} X_i\Big) \arrow[d] \\
\bigoplus_{i=0}^{\infty} X \arrow[r, "\mathrm{id}-\mathrm{shift}"'] & \bigoplus_{i=0}^{\infty} X \arrow[r] & X \arrow[r, "0"] & T\Big(\bigoplus_{i=0}^{\infty} X\Big).
\end{tikzcd}$$

Applying H^n, we obtain a commutative diagram

$$\begin{tikzcd}
\bigoplus_i \mathrm{H}^n(X_i) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_i \mathrm{H}^n(X_i) \arrow[r] \arrow[d, "\simeq"] & \mathrm{H}^n(\operatorname{hocolim}_i X_i) \arrow[r] \arrow[d, "\mathrm{H}^n(f)"] & \bigoplus_i \mathrm{H}^{n+1}(X_i) \arrow[r] \arrow[d, "\simeq"] & \bigoplus_i \mathrm{H}^{n+1}(X_i) \arrow[d, "\simeq"] \\
\bigoplus_i \mathrm{H}^n(X) \arrow[r] & \bigoplus_i \mathrm{H}^n(X) \arrow[r] & \mathrm{H}^n(X) \arrow[r, "0"'] & \bigoplus_i \mathrm{H}^{n+1}(X) \arrow[r] & \bigoplus_i \mathrm{H}^{n+1}(X).
\end{tikzcd}$$

Now the five lemma shows that $\mathrm{H}^n(f)$ is an isomorphism as desired. □

Example 17.10. Let $\mathcal{A}$ be an abelian category and $X \in \mathsf{D}(\mathcal{A})$.

(i) Suppose that $\mathcal{A}$ is AB4 (infinite direct sums exist and are exact). Then the natural maps

$$\operatorname*{hocolim}_i \tau^{\leq i} X \xrightarrow{\sim} X \qquad \text{and} \qquad \operatorname*{hocolim}_i \sigma^{\geq -i} X \xrightarrow{\sim} X$$

are isomorphisms in $\mathsf{D}(\mathcal{A})$.

(ii) Suppose that $\mathcal{A}$ is AB4* (infinite products exist and are exact). Then the natural maps

$$X \xrightarrow{\sim} \operatorname*{holim}_i \tau^{\geq -i} X \qquad \text{and} \qquad X \xrightarrow{\sim} \operatorname*{holim}_i \sigma^{\leq i} X$$

are isomorphisms in $\mathsf{D}(\mathcal{A})$.

We can now prove the analog of Theorem 15.6 for unbounded derived categories:

Theorem 17.11. *Let $\mathcal{A}$ be an abelian category which has enough injectives and is AB4*. Denote by $\mathcal{I} \subseteq \mathcal{A}$ the full subcategory of injective objects. Then*

(i) The functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is a Bousfield localization. More precisely, let $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \subseteq \mathsf{K}(\mathcal{A})$ be the smallest triangulated subcategory containing $\mathcal{I}$ and which is closed under arbitrary products. Then the restriction $Q|_{\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})}\colon \mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \xrightarrow{\sim} \mathsf{D}(\mathcal{A})$ is an equivalence of triangulated categories. We denote by

$$\mathbf{i}\colon \mathsf{D}(\mathcal{A}) \xrightarrow{\sim} \mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \subseteq \mathsf{K}(\mathcal{A})$$

a right adjoint of Q which factors through $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$.

(ii) Let $F\colon \mathcal{A} \to \mathcal{B}$ be an additive functor of abelian categories. Then the right derived functor $\mathrm{R}F$ is computed as the composition

$$\mathsf{D}(\mathcal{A}) \xrightarrow{\mathbf{i}} \mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \xrightarrow{\mathsf{K}F|_{\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})}} \mathsf{K}(\mathcal{B}) \xrightarrow{Q_{\mathcal{B}}} \mathsf{D}(\mathcal{B}).$$

Proof. We prove the following statements:

(a) Every object $I \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ is local with respect to quasi-isomorphisms or, equivalently, $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) = 0$ for every acyclic complex $X \in \mathsf{K}(\mathcal{A})$.

(b) $\mathsf{K}^+(\mathcal{I}) \subseteq \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$.

(c) For every $X \in \mathsf{K}(\mathcal{A})$ there exists a quasi-isomorphism $X \to I$ with $I \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$.

Once this is done, the rest of the argument follows exactly as in Theorem 15.6. For part (a), let $\mathcal{L} \subseteq \mathsf{K}(\mathcal{A})$ be the full subcategory spanned by all objects I such that $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) = 0$ for all acyclic complexes $X \in \mathsf{K}(\mathcal{A})$. We first show that $\mathcal{L}$ is a triangulated subcategory. It is obvious that $\mathcal{L}[1] = \mathcal{L}$. Let now $I \to I' \to I'' \to I[1]$ be a distinguished triangle in $\mathsf{K}(\mathcal{A})$ such that $I, I' \in \mathcal{L}$. We claim that $I'' \in \mathcal{L}$. Indeed, let $X \in \mathsf{K}(\mathcal{A})$ be an arbitrary acyclic complex. Since $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, -)$ is cohomological (Proposition 4.7), we obtain an exact sequence

$$0 = \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I') \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I'') \to \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I[1]) = 0,$$

from which we deduce $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I'') = 0$. Hence, $I'' \in \mathcal{L}$, which shows that $\mathcal{L} \subseteq \mathcal{K}(\mathcal{A})$ is a triangulated subcategory. Since $\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, -)$ preserves arbitrary products, it is clear that $\mathcal{L}$ is closed under arbitrary products. It follows that $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ is contained in $\mathcal{L}$, which verifies (a).

Let us now prove (b). We first show $\mathsf{K}^b(\mathcal{I}) \subseteq \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. Note that every $I \in \mathsf{K}^b(\mathcal{I})$ is isomorphic to the mapping cone of the map $\sigma^{\leq n} I[-1] \xrightarrow{d_n} \sigma^{>n} I$. Hence by induction on the number of non-zero terms in I we deduce $I \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. Now, for every $I \in \mathsf{K}^+(\mathcal{I})$ the map $I \to \mathrm{holim}_n\, \sigma^{\leq n} I$ is a quasi-isomorphism by Example 17.10. Since I is local by Theorem 15.6 and $\mathrm{holim}_n\, \sigma^{\leq n} I$ is local by (a), we deduce from Theorem 9.4 that the map is an isomorphism. Hence $I \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$.

For part (c), observe that for every $n \geq 0$ there exists a quasi-isomorphism $\tau^{\geq -n} X \to I_n$ with $I_n \in \mathsf{K}^+(\mathcal{I})$, by Theorem 15.6(ii). Since $I_n \in \mathsf{K}_{\mathrm{hinj}}(A)$ is local by (a), the map

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(I_{n+1}, I_n) \xrightarrow{\sim} \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(\tau^{\geq -n-1} X, I_n)$$

given by precomposition with $\tau^{\geq -n-1} X \to I_{n+1}$ is an isomorphism. Hence, there exists a map $f_{n+1} \colon I_{n+1} \to I_n$ making the diagram

$$\begin{tikzcd}
\tau^{\geq -n-1} X \arrow[r] \arrow[d, "\mathrm{qis}"'] & \tau^{\geq -n} X \arrow[d, "\mathrm{qis}"] \\
I_{n+1} \arrow[r, dashed, "f_{n+1}"'] & I_n
\end{tikzcd}$$

commutative. We obtain a sequence of maps $\cdots \xrightarrow{f_3} I_2 \xrightarrow{f_2} I_1 \xrightarrow{f_1} I_0$ such that each I_n lies in $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. We deduce that $I \coloneqq \mathrm{holim}_i\, I_i \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. Consider now the commutative diagram

$$\begin{tikzcd}
\prod_{i=0}^{\infty} \tau^{\geq -i} X[-1] \arrow[r, "\mathrm{id}-\mathrm{shift}"] \arrow[d, "\mathrm{qis}"'] & \prod_{i=0}^{\infty} \tau^{\geq -i} X[-1] \arrow[r] \arrow[d, "\mathrm{qis}"'] & X \arrow[r] \arrow[d, dashed] & \prod_{i=0}^{\infty} \tau^{\geq -i} X \arrow[d, "\mathrm{qis}"] \\
\prod_{i=0}^{\infty} I_i[-1] \arrow[r, "\mathrm{id}-\mathrm{shift}"'] & \prod_{i=0}^{\infty} I_i[-1] \arrow[r] & I \arrow[r] & \prod_{i=0}^{\infty} I_i
\end{tikzcd}$$

where the rows are distinguished triangles by definition of homotopy limit and by Example 17.10(ii). By (S6) and Proposition 12.3(i), the dashed arrow exists and is a quasi-isomorphism. This finishes the proof. $\square$

Remark 17.12. The assumption that $\mathcal{A}$ be AB4* covers a large class of abelian categories of interest, like the category of modules over a ring. However, there are also many interesting abelian categories in which infinite products are not exact. This generally happens for various categories of sheaves on a topological space or a scheme.

It turns out that basically all abelian categories of interest are Grothendieck, *i.e.*, they contain a generator, all colimits exist, and the formation of filtered colimits is exact. There is an analog of Theorem 17.11 for Grothendieck abelian categories, which we will explain in Theorem 24.10.

Chapter 5

Spectral Sequences

§18. Definition of spectral sequences

Doing computations in the derived category is usually very easy and conceptual. It is often necessary to extract information in terms of the abelian level. We have two examples in mind:

- Let $X \in \mathsf{C}^*(\mathcal{A})$ be a complex and $\mathrm{R}F\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ a derived functor. Can we compute $\mathrm{H}^n(\mathrm{R}F(X))$ in terms of the $\mathrm{R}^n F$ and $\mathrm{H}^n(X)$ (or even X^n)?
- Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors and $X \in \mathsf{K}^+(\mathcal{A})$. Can we compute $\mathrm{H}^n(\mathrm{R}G(\mathrm{R}F(X)))$ in terms of the higher derived functors of G and F?

The computational tool that allows us to treat these problems is called a *spectral sequence*. We first point out that the second of the above questions is related to the computation of the derived functor of a composition of two functors, namely, we have the following result:

Proposition 18.1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C}$ be abelian categories, let $*, \dagger \in \{\varnothing, b, +, -\}$. Let $F\colon \mathsf{K}^*(\mathcal{A}) \to \mathsf{K}^\dagger(\mathcal{B})$ and $G\colon \mathsf{K}^\dagger(\mathcal{B}) \to \mathsf{K}(\mathcal{C})$ be exact functors. Suppose that the following conditions are satisfied:*

(a) There exist full subcategories $\mathcal{L} \subseteq \mathsf{K}^*(\mathcal{A})$ and $\mathcal{M} \subseteq \mathsf{K}^\dagger(\mathcal{B})$ satisfying the hypotheses of Theorem 15.1 for F and G, respectively.

(b) $F(\mathcal{L}) \subseteq \mathcal{M}$.

Then the canonical map

$$\mathrm{R}(G \circ F) \xrightarrow{\sim} \mathrm{R}G \circ \mathrm{R}F$$

is an isomorphism of functors $\mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{C})$.

Proof. It follows from (a), (b) and Theorem 15.1 that the right derived functors $\mathrm{R}F\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}^\dagger(\mathcal{B})$, $\mathrm{R}G\colon \mathsf{D}^\dagger(\mathcal{B}) \to \mathsf{D}(\mathcal{C})$, and $\mathrm{R}(G \circ F)\colon \mathsf{D}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{C})$ exist. We obtain from the universal property of the derived functor a unique map $\alpha\colon \mathrm{R}(G \circ F) \longrightarrow \mathrm{R}G \circ \mathrm{R}F$ making the diagram

$$\begin{tikzcd}
Q_{\mathcal{C}}GF \arrow[rr] \arrow[d, equal] & & \mathrm{R}(G \circ F)Q_{\mathcal{A}} \arrow[d, dashed, "\alpha Q_{\mathcal{A}}"] \\
Q_{\mathcal{C}}GF \arrow[r] & \mathrm{R}(G)Q_{\mathcal{B}}F \arrow[r] & \mathrm{R}(G)\mathrm{R}(F)Q_{\mathcal{A}}
\end{tikzcd}$$

of functors $\mathsf{K}^*(\mathcal{A}) \to \mathsf{D}(\mathcal{C})$ commutative. Note that, applied to objects of $\mathcal{L}$, the horizontal maps in the diagram are isomorphisms, because $F(\mathcal{L}) \subseteq \mathcal{M}$. Hence, for all $L \in \mathcal{L}$ the map $\alpha_{Q_\mathcal{A} L}$ is an isomorphism. Now, for any $X \in \mathsf{K}^*(\mathcal{A})$, choose a quasi-isomorphism $X \to L$ with $L \in \mathcal{L}$. Then we have a commutative diagram

$$\begin{tikzcd} \mathrm{R}(G \circ F)(Q_\mathcal{A} X) \arrow[r, "\alpha_{Q_\mathcal{A} X}"] \arrow[d, "\sim"'] & (\mathrm{R}G \circ \mathrm{R}F)(Q_\mathcal{A} X) \arrow[d, "\sim"] \\ \mathrm{R}(G \circ F)(Q_\mathcal{A} L) \arrow[r, "\sim", "\alpha_{Q_\mathcal{A} L}"'] & (\mathrm{R}G \circ \mathrm{R}F)(Q_\mathcal{A} L), \end{tikzcd}$$

which implies that $\alpha_{Q_\mathcal{A} X}$ is an isomorphism. As $Q_\mathcal{A}$ is essentially surjective, we deduce that α is an isomorphism as desired. □

We now introduce the notion of a spectral sequence.

Definition 18.2. Let $\mathcal{A}$ be an abelian category. A *spectral sequence* consists of the following data:

(a) a family $\{E_r^{pq}\}_{r \geq a, p,q \in \mathbb{Z}}$ of objects of $\mathcal{A}$ (for some $a \geq 0$);

(b) differentials $d_r^{pq} \colon E_r^{pq} \to E_r^{p+r,q-r+1}$, *i.e.*, maps satisfying $d_r^{p+r,q-r+1} \circ d_r^{pq} = 0$ for all p, q, r;

(c) isomorphisms $\alpha_r^{pq} \colon E_{r+1}^{pq} \xrightarrow{\sim} \operatorname{Ker}(d_r^{pq}) / \operatorname{Im}(d_r^{p-r,q+r-1})$.

The *total degree* of E_r^{pq} is defined to be $p+q$. We call $E_r = \{E_r^{pq}\}_{p,q}$ the *r-th page* of the spectral sequence.

A morphism $f \colon E \to E'$ of spectral sequences consists of maps $f_r^{pq} \colon E_r^{pq} \to E_r'^{pq}$ commuting with the differentials (*i.e.*, $d_r'^{pq} f_r^{pq} = f_r^{p+r,q-r+1} d_r^{pq}$) such that the diagrams

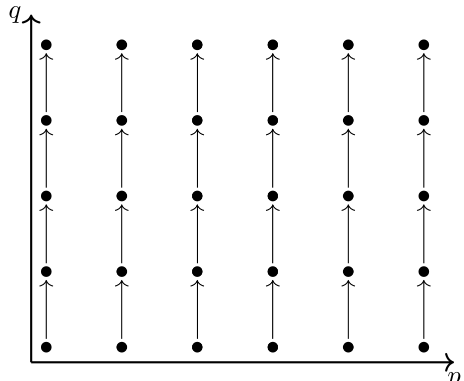
$$\begin{tikzcd} E_{r+1}^{pq} \arrow[r, "f_{r+1}^{pq}"] \arrow[d, "\alpha_r^{pq}"', "\sim"] & E_{r+1}'^{pq} \arrow[d, "\alpha_r'^{pq}", "\sim"'] \\ \operatorname{Ker}(d_r^{pq}) / \operatorname{Im}(d_r^{p-r,q+r-1}) \arrow[r, "f_r^{pq}"'] & \operatorname{Ker}(d_r'^{pq}) / \operatorname{Im}(d_r'^{p-r,q+r-1}) \end{tikzcd}$$

commute for all r, p, q.

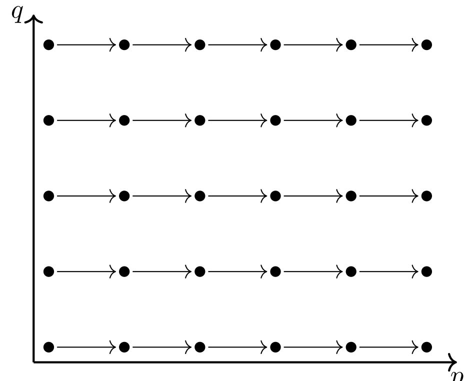
We point out the following immediate observations:

- E_{r+1}^{pq} is a subquotient of E_r^{pq}.
- the differentials d_r^{pq} increase the total degree by 1.

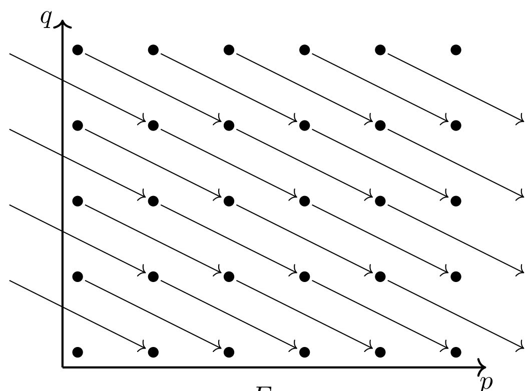
We illustrate the spectral sequence for its first pages:



E_0



E_1



E_2

Definition 18.3. A spectral sequence $E = \{E_r^{pq}\}_{r \geq a, p, q}$ is called *bounded* if for all n there are only finitely many (p, q) with $p + q = n$ and $E_a^{pq} \neq 0$. Note that in this case the differentials d_r^{pq} vanish for fixed p, q and $r \gg 0$, and hence $E_r^{pq} \cong E_{r+1}^{pq} \cong \cdots$. We denote by $E_\infty^{pq} = E_r^{pq}$ this stable value of E_r^{pq}.

We say that a bounded spectral sequence E *converges* to $H = \{H^n\}_{n \in \mathbb{Z}}$ if each $H^n \in \mathcal{A}$ admits a *finite* descending filtration

$$0 = F^s H^n \subseteq F^{s-1} H^n \subseteq \cdots \subseteq F^t H^n = H^n$$

such that $E_\infty^{pq} \cong F^p H^{p+q} / F^{p+1} H^{p+q}$ for all p, q. We denote convergence by writing

$$E_a^{pq} \implies H^{p+q}.$$

Example 18.4. (a) Let $\mathcal{A}$ be an abelian category. Every *first quadrant* spectral sequence $E = \{E_r^{pq}\}_{p,q \geq 0}$ is bounded; every *third quadrant* spectral sequence $E = \{E_r^{pq}\}_{p,q \leq 0}$ is bounded.

(b) Let G be a finite group and $N \trianglelefteq G$ a normal subgroup. The *Lyndon–Hochschild–Serre spectral sequence* is a converging first quadrant spectral sequence

$$E_2^{pq} = \mathrm{H}^p(G/N, \mathrm{H}^q(N, A)) \implies \mathrm{H}^{p+q}(G, A)$$

for any G-representation A. It is a special case of the Grothendieck spectral sequence, which we will prove later (see Theorem 20.10).

§19. Construction of spectral sequences

Classically, spectral sequences were constructed from filtered chain complexes. We will follow the exposition in [Ben91], but will work in slightly greater generality.

We start by introducing the notion of an "exact couple", which is the abstract machine which constructs for us spectral sequences.

Definition 19.1 (Massey)**.** Let $\mathcal{A}$ be an abelian category. An *exact couple* is a tuple (D, E, i, j, k) consisting of objects $D, E \in \mathcal{A}$ and morphisms $D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{k} D$ such that

$$\operatorname{Ker}(i) = \operatorname{Im}(k), \qquad \operatorname{Ker}(j) = \operatorname{Im}(i), \qquad \operatorname{Ker}(k) = \operatorname{Im}(j).$$

Since $k \circ j = 0$, the map $d := j \circ k \colon E \to E$ defines a differential on E. We define its cohomology object to be

$$\mathrm{H}(E, d) := \operatorname{Ker}(d)/\operatorname{Im}(d).$$

We visualize an exact couple as a triangle

$$\begin{tikzcd} D \arrow[rr, "i"] & & D \arrow[dl, "j"] \\ & E. \arrow[ul, "k"] & \end{tikzcd}$$

Lemma 19.2. *Let $\mathcal{A}$ be an abelian category and (D, E, i, j, k) an exact couple. We put*

$$\begin{aligned} D' &:= \operatorname{Im}(i) \subseteq D, \\ E' &:= \mathrm{H}(E, d) \quad \textit{as a subquotient of } E, \end{aligned}$$

and define maps $D' \xrightarrow{i'} D' \xrightarrow{j'} E' \xrightarrow{k'} D'$ by the commutativity of the diagram

$$\begin{tikzcd} & E \arrow[r, two heads, "d"] \arrow[d, "k"] & \operatorname{Im}(d) \arrow[d, hook] & & \\ \operatorname{Im}(i) \arrow[r, hook] \arrow[d, equal] & D \arrow[r, "j"] \arrow[d, two heads, "i"] & \operatorname{Ker}(d) \arrow[r, "k"] \arrow[d, two heads] & \operatorname{Ker}(j) \arrow[d, equal] & = \operatorname{Im}(i) \\ D' \arrow[r, dashed, "i'"'] & D' \arrow[r, dashed, "j'"'] & E' \arrow[r, dashed, "k'"'] & D'. & \end{tikzcd}$$

Then (D', E', i', j', k') defines an exact couple, called the derived couple *of (D, E, i, j, k).*

Proof. From the construction it is clear that $\operatorname{Im}(i') \subseteq \operatorname{Ker}(j')$, $\operatorname{Im}(j') \subseteq \operatorname{Ker}(k')$, and $\operatorname{Im}(k') \subseteq \operatorname{Ker}(i')$. A diagram chase shows that these inclusions are epimorphisms, hence isomorphisms. □

Lemma 19.3. *Let (D, E, i, j, k) be an exact couple with differential $d = j \circ k$. For $r \geq 2$ we denote by $(D_r, E_r, i_r, j_r, k_r)$ the $(r-1)$-th derived couple with differential $d_r = j_r \circ k_r$. Put $Z_r := k^{-1}(\operatorname{Im}(i^{r-1}))$ and $B_r := j(\operatorname{Ker}(i^{r-1}))$ as subobjects of E. Then there are canonical isomorphisms*

$$E_r \cong Z_r/B_r, \qquad B_r/B_{r-1} \cong \operatorname{Im}(d_{r-1}), \qquad Z_r/B_{r-1} \cong \operatorname{Ker}(d_{r-1}).$$

Proof. We prove the claim by induction on r. The case $r = 1$ is trivial. So let $r = 2$. Then

$$\begin{aligned} B_2 &= j(\operatorname{Ker}(i)) = j(\operatorname{Im}(k)) = \operatorname{Im}(j \circ k) = \operatorname{Im}(d), \\ Z_2 &= k^{-1}(\operatorname{Im}(i)) = k^{-1}(\operatorname{Ker}(j)) = \operatorname{Ker}(j \circ k) = \operatorname{Ker}(d), \end{aligned}$$

so that $Z_2/B_2 \cong E_2$. Let now $r > 2$ and suppose that the claim is true for $r - 1$ and every exact couple. Consider the commutative diagram

$$\begin{tikzcd} D \arrow[r, "j"] \arrow[d, "i"'] & Z_r \arrow[r, "k"] \arrow[d, two heads, "\pi"] & \operatorname{Im}(i) \arrow[d, equal] \\ D_2 \arrow[r, "j_2"'] & Z_r/B_2 \arrow[r, "k_2"'] & D_2. \end{tikzcd}$$

We compute

$$\begin{aligned} \widetilde{Z}_{r-1} &:= k_2^{-1}(\operatorname{Im}(i_2^{r-2})) = \pi k^{-1}\big(\operatorname{Im}(i^{r-1})\big) = \pi(Z_r) = Z_r/B_2, \\ \widetilde{B}_{r-1} &:= j_2\big(\operatorname{Ker}(i_2^{r-2})\big) = j_2 i\big(\operatorname{Ker}(i^{r-1})\big) = \pi j\big(\operatorname{Ker}(i^{r-1})\big) = \pi(B_r) = B_r/B_2. \end{aligned}$$

Hence, by the induction hypothesis for $(D_2, E_2, i_2, j_2, k_2)$, we deduce

$$\begin{aligned} Z_r/B_r &\cong \frac{Z_r/B_{r-1}}{B_r/B_{r-1}} \cong \operatorname{Ker}(d_{r-1})/\operatorname{Im}(d_{r-1}) = E_r, \\ B_r/B_{r-1} &\cong \frac{B_r/B_2}{B_{r-1}/B_2} \cong \widetilde{B}_{r-1}/\widetilde{B}_{r-2} \cong \operatorname{Im}(\widetilde{d}_{r-2}) = \operatorname{Im}(d_{r-1}), \\ Z_r/B_{r-1} &\cong \frac{Z_r/B_2}{B_{r-1}/B_2} \cong \widetilde{Z}_{r-1}/\widetilde{B}_{r-2} \cong \operatorname{Ker}(\widetilde{d}_{r-2}) = \operatorname{Ker}(d_{r-1}). \end{aligned}$$ □

Remark 19.4. Classically, spectral sequences are obtained from a filtration on a chain complex. Recall that if $C \in \mathsf{C}(\mathcal{A})$ is a complex with descending filtration

$$\cdots \subseteq F^{p+1}C \subseteq F^pC \subseteq F^{p-1}C \subseteq \cdots \subseteq C$$

we may look at the associated graded complexes $\operatorname{gr}^p C := F^pC/F^{p+1}C$. The idea of the spectral sequence associated with $F^\bullet C$ is to approximate the cohomology of C with the cohomologies of the associated graded complexes $\operatorname{gr}^p C$: We then want to build a convergent spectral sequence

$$E_2^{p,q} = \mathrm{H}^{p+q}(\operatorname{gr}^p C) \implies \mathrm{H}^{p+q}(C).$$

The construction requires us to also consider the complexes F^pC/F^qC for $q \geq p$. Now, it is sometimes more useful to construct the necessary data directly in the derived category. This motivates the following definition:

Definition 19.5. Let $\mathcal{A}$ be an abelian category. Let $\mathcal{I}_0 = \mathbb{Z} \cup \{\pm\infty\}$ with the obvious partial ordering and consider partially ordered set

$$\mathcal{I} = \left\{(p,q) \in \mathcal{I}_0^2 \,\middle|\, p \leq q\right\}$$

with partial ordering $(p,q) \leq (r,s)$ if $p \leq r$ and $q \leq s$. A *(descending) filtered object* in $\mathsf{D}(\mathcal{A})$ is a functor

$$X \colon \mathcal{I}^{\mathrm{op}} \to \mathsf{D}(\mathcal{A})$$

together with maps $\delta_{p,q,r} \colon X(p,q) \to X(q,r)[1]$ for all $p \leq q \leq r$ satisfying the following conditions:

- $X(p,p) = 0$ for all p.
- For all $p \leq q \leq r$ the triangle $X(q,r) \to X(p,r) \to X(p,q) \xrightarrow{\delta_{p,q,r}} X(q,r)[1]$ is distinguished.
- For all $p \leq q \leq r \leq s$ the induced diagram

$$\begin{array}{ccccccc} X(q,s) & \longrightarrow & X(p,s) & \longrightarrow & X(p,q) & \xrightarrow{\delta_{p,q,s}} & X(q,s)[1] \\ \downarrow & & \downarrow & & \| & & \downarrow \\ X(q,r) & \longrightarrow & X(p,r) & \longrightarrow & X(p,q) & \xrightarrow[\delta_{p,q,r}]{} & X(q,r)[1] \end{array} \tag{19.1}$$

  is a morphism of (distinguished) triangles.

In particular, if we define $X(p) \coloneqq X(p,\infty)$, then we obtain a chain of maps

$$\cdots \to X(p+1) \to X(p) \to X(p-1) \to \cdots \to X(-\infty),$$

in $\mathsf{D}(\mathcal{A})$, which we view as a descending filtration of $X(-\infty)$. Moreover, we recover the terms $X(p,q)$ via the distinguished triangles

$$X(q) \to X(p) \to X(p,q) \to X(q)[1]$$

for all $p \leq q$ in $\mathbb{Z}$.

Example 19.6. Let $\mathcal{A}$ be an abelian category and $C \in \mathsf{C}(\mathcal{A})$ a complex together with a filtration

$$0 \subseteq \cdots \subseteq F^{p+1}C \subseteq F^pC \subseteq F^{p-1}C \subseteq \cdots \subseteq C$$

by subcomplexes. Putting $F^{\infty}C \coloneqq 0$ and $F^{-\infty}C \coloneqq C$, we obtain a filtered object of $\mathsf{D}(\mathcal{A})$ via $X(p,q) \coloneqq Q(F^pC/F^qC)$ for all $p \leq q$. Indeed, observe that for all $p \leq q \leq r \leq s$ we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^qC/F^sC & \longrightarrow & F^pC/F^sC & \longrightarrow & F^pC/F^qC & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & F^qC/F^rC & \longrightarrow & F^pC/F^rC & \longrightarrow & F^pC/F^qC & \longrightarrow & 0, \end{array}$$

and then Example 12.4 gives the map of distinguished triangles (19.1).

Example 19.7. Let $\mathcal{A}$ be an abelian category. As a special case of Example 19.6 we obtain the following filtered objects in $\mathsf{D}(\mathcal{A})$:

(a) Let $C \in \mathsf{C}(\mathcal{A})$ with filtration given by $F^pC \coloneqq \tau^{\leq -p}C$. For the induced filtered object X in $\mathsf{D}(A)$ we then have
$$X(p,q) = Q(\tau^{>-q}\tau^{\leq -p}C)$$
in $\mathsf{D}(\mathcal{A})$, whose cohomologies are concentrated in degrees $[-q+1,\dots,-p]$. In particular, we have $X(p,p+1) = \mathrm{H}^{-p}(C)[p]$.

(b) Let $C \in \mathsf{C}(\mathcal{A})$ with filtration given by $F^pC \coloneqq \sigma^{\geq p}C$. For the induced filtered object X in $\mathsf{D}(\mathcal{A})$ we then have
$$X(p,q) = Q(\sigma^{<q}\sigma^{\geq p}C)$$
in $\mathsf{D}(\mathcal{A})$, whose cohomologies are concentrated in degrees $[p,\dots,q-1]$. In particular, we have $X(p,p+1) = C^p[-p]$.

Example 19.8. If X is a filtered object of $\mathsf{D}(\mathcal{A})$ and $F\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ is an exact functor between derived categories, then $F \circ X$ is a filtered object of $\mathsf{D}(\mathcal{B})$.

Using exact couples and their derived couples we can now construct spectral sequences:

Construction 19.9. Let $\mathcal{A}$ be an abelian category and $X(-,-)$ a filtered object of $\mathsf{D}(\mathcal{A})$. The long exact sequences
$$\cdots \to \mathrm{H}^n(X(p+1)) \xrightarrow{i_1} \mathrm{H}^n(X(p)) \xrightarrow{j_1} \mathrm{H}^n(X(p,p+1)) \xrightarrow{k_1} \mathrm{H}^{n+1}(X(p+1)) \to \cdots$$
fit into a large diagram
$$\begin{tikzcd}
 & {} \arrow[d] & & {} \arrow[d] & & \\
{} \arrow[r, "k_1"] & \mathrm{H}^{n-1}(X(p+1)) \arrow[r] \arrow[d, "i_1"] & \mathrm{H}^{n-1}(X(p+1,p+2)) \arrow[r] & \mathrm{H}^n(X(p+2)) \arrow[r] \arrow[d] & \mathrm{H}^n(X(p+2,p+3)) \arrow[r] & {} \\
{} \arrow[r] & \mathrm{H}^{n-1}(X(p)) \arrow[r, "j_1"] \arrow[d] & \mathrm{H}^{n-1}(X(p,p+1)) \arrow[r, "k_1"] & \mathrm{H}^n(X(p+1)) \arrow[r] \arrow[d, "i_1"] & \mathrm{H}^n(X(p+1,p+2)) \arrow[r] & {} \\
{} \arrow[r] & \mathrm{H}^{n-1}(X(p-1)) \arrow[r] \arrow[d] & \mathrm{H}^{n-1}(X(p-1,p)) \arrow[r] & \mathrm{H}^n(X(p)) \arrow[r, "j_1"] \arrow[d] & \mathrm{H}^n(X(p,p+1)) \arrow[r, "k_1"] & {} \\
 & {} & & {} & &
\end{tikzcd}$$

We now put
$$E_1^{pq} \coloneqq \mathrm{H}^{p+q}(X(p,p+1)), \qquad D_1^{pq} \coloneqq \mathrm{H}^{p+q}(X(p))$$
and visualize them in a triangular shape,
$$\begin{tikzcd}
D_1^{\bullet,\bullet} \arrow[rr, "i_1"] & & D_1^{\bullet,\bullet} \arrow[dl, "j_1"] \\
 & E_1^{\bullet,\bullet}, \arrow[ul, "k_1"] &
\end{tikzcd}$$

where

$$\deg(i_1) = (-1,1), \qquad \deg(j_1) = (0,0), \qquad \deg(k_1) = (1,0),$$

and

$$\mathrm{Ker}(i_1) = \mathrm{Im}(k_1), \qquad \mathrm{Ker}(j_1) = \mathrm{Im}(i_1), \qquad \mathrm{Ker}(k_1) = \mathrm{Im}(j_1).$$

We define the exact couple $(D_r, E_r, i_r, j_r, k_r)$ as the $(r-1)$-th derived couple of of (D_1, E_1). The sequence $\{(E_r, d_r)\}_{r\geq 1}$ is the *spectral sequence* of the original exact couple (D_1, E_1). We keep track of the double grading

$$\begin{aligned}
\deg(i_r) &= \deg(i_{r-1}),\\
\deg(j_r) &= \deg(j_{r-1}) - \deg(i_{r-1}),\\
\deg(k_r) &= \deg(k_{r-1}),\\
\deg(d_r) &= \deg(j_r) + \deg(k_r),
\end{aligned}$$

so that, by induction,

$$\begin{aligned}
\deg(i_r) &= (-1,1),\\
\deg(j_r) &= (r-1, -r+1),\\
\deg(k_r) &= (1,0),\\
\deg(d_r) &= (r, -r+1).
\end{aligned}$$

Fix p, q. Note that $D^{pq}_{r+1} \subseteq D^{pq}_r \subseteq D^{pq}_1$ for all r, and we put $D^{pq}_\infty \coloneqq \bigcap_{r\geq 1} D^{pq}_r$. Each E^{pq}_r is a subquotient of E^{pq}_1, hence we find subobjects $B^{pq}_r, Z^{pq}_r \subseteq E^{pq}_1$ such that

$$E^{pq}_r = Z^{pq}_r / B^{pq}_r.$$

By the construction and Lemma 19.3 we have

$$E^{pq}_1 = Z^{pq}_1 \supseteq Z^{pq}_2 \supseteq \cdots \supseteq Z^{pq}_r \supseteq \cdots \supseteq B^{pq}_r \supseteq \cdots \supseteq B^{pq}_2 \supseteq B^{pq}_1 = 0,$$

where $Z^{pq}_r / B^{pq}_{r-1} = \mathrm{Ker}(d^{pq}_{r-1})$ and $B^{pq}_r / B^{pq}_{r-1} = \mathrm{Im}(d^{p-r,q+r-1}_{r-1})$. We put $B^{pq}_\infty \coloneqq \bigcup_r B^{pq}_r$ and $Z^{pq}_\infty \coloneqq \bigcap_r Z^{pq}_r$. Note that $B^{pq}_\infty \subseteq Z^{pq}_\infty$, so that the object

$$E^{pq}_\infty \coloneqq Z^{pq}_\infty / B^{pq}_\infty$$

is well-defined. *We make the following simplifying assumptions:*

($*$) For fixed p, q and $n \gg 0$, we have

$$\begin{aligned}
&\mathrm{H}^{p+q}(X(p+n)) = 0, \qquad \text{and}\\
&\mathrm{H}^{p+q}(X(p-n)) \xrightarrow{\sim} \mathrm{H}^{p+q}(X(-\infty))
\end{aligned}$$

is an isomorphism.

Proposition 19.10. *The following equalities hold:*

(i)
$$\begin{aligned} Z_n^{pq} &= \mathrm{Im}\big(\mathrm{H}^{p+q}(X(p,p+n)) \to \mathrm{H}^{p+q}(X(p,p+1))\big) \\ &= \mathrm{Ker}\big(\mathrm{H}^{p+q}(X(p,p+1)) \xrightarrow{\partial} \mathrm{H}^{p+q+1}(X(p+1,p+n))\big). \end{aligned}$$

(ii)
$$\begin{aligned} B_n^{pq} &= \mathrm{Im}\big(\mathrm{H}^{p+q-1}(X(p-n+1,p)) \xrightarrow{\partial} \mathrm{H}^{p+q}(X(p,p+1))\big) \\ &= \mathrm{Ker}\big(\mathrm{H}^{p+q}(X(p,p+1)) \to \mathrm{H}^{p+q}(X(p-n+1,p+1))\big). \end{aligned}$$

Remark 19.11. Observe that the assumption $(*)$ and Proposition 19.10 imply that, for fixed p,q and $n \gg 0$:

$$\begin{aligned} Z_n^{pq} &= Z_{n+1}^{pq} = \cdots = Z_\infty^{pq} = \mathrm{Im}\big(\mathrm{H}^{p+q}(X(p)) \to E_1^{pq}\big) \\ B_n^{pq} &= B_{n+1}^{pq} = \cdots = B_\infty^{pq} = \mathrm{Im}\big(\mathrm{H}^{p+q-1}(X(-\infty,p)) \to E_1^{pq}\big), \end{aligned}$$

where $E_1^{pq} = \mathrm{H}^{p+q}(X(p,p+1))$.

Proof of Proposition 19.10. For part (i), we consider the diagram

$$\begin{tikzcd} & & & X(p+n) \arrow[d, "(i_1)^{n-1}"] \\ X(p+1) \arrow[r] \arrow[d] & X(p) \arrow[r] \arrow[d] & X(p,p+1) \arrow[r, "k_1"] \arrow[d, equal] & X(p+1)[1] \arrow[d, "\tilde{\jmath}_1"] \\ X(p+1,p+n) \arrow[r] & X(p,p+n) \arrow[r] & X(p,p+1) \arrow[r, "\partial"'] & X(p+1,p+n)[1] \end{tikzcd}$$

which commutes by (19.1). Applying H^{p+q} to the bottom row, we obtain an exact sequence

$$\mathrm{H}^{p+q}(X(p,p+n)) \xrightarrow{\alpha} \mathrm{H}^{p+q}(X(p,p+1)) \xrightarrow{\partial} \mathrm{H}^{p+q+1}(X(p+1,p+n))$$

and the claim is that $Z_n^{pq} = \mathrm{Im}(\alpha) = \mathrm{Ker}(\partial)$. We consider the following commutative diagram

$$\begin{tikzcd} & \mathrm{H}^{p+q+1}(X(p+n)) \arrow[d, "(i_1)^{n-1}"] \\ \mathrm{H}^{p+q}(X(p,p+1)) \arrow[r, "k_1"] \arrow[dr, bend right, "\partial"'] & \mathrm{H}^{p+q+1}(X(p+1)) \arrow[d, "\tilde{\jmath}_1"] \\ & \mathrm{H}^{p+q+1}(X(p+1,p+n)). \end{tikzcd}$$

By Lemma 19.3 we have

$$Z_n^{pq} = k_1^{-1}(\mathrm{Im}(i_1^{n-1})) = k_1^{-1}(\mathrm{Ker}(\tilde{\jmath}_1)) = \mathrm{Ker}(\tilde{\jmath}_1 \circ k_1) = \mathrm{Ker}(\partial) = \mathrm{Im}(\alpha)$$

as desired.

Part (ii) works similarly. We instead consider the morphism of distinguished triangles

$$\begin{tikzcd} X(p) \arrow[r, "(i_1)^{n-1}"] \arrow[d] & X(p-n+1) \arrow[r] \arrow[d] & X(p-n+1,p) \arrow[r, "\tilde{k}_1"] \arrow[d, equal] & X(p)[1] \arrow[d, "j_1"] \\ X(p,p+1) \arrow[r] & X(p-n+1,p+1) \arrow[r] & X(p-n+1,p) \arrow[r, "\partial"'] & X(p,p+1)[1]. \end{tikzcd}$$

Applying H^{p+q-1} to the bottom row, we obtain an exact sequence

$$\mathrm{H}^{p+q-1}(X(p-n+1,p)) \xrightarrow{\partial} \mathrm{H}^{p+q}(X(p,p+1)) \xrightarrow{\beta} \mathrm{H}^{p+q}(X(p,p+1))$$

and the claim is that $B_n^{pq} = \mathrm{Im}(\partial) = \mathrm{Ker}(\beta)$. We consider the following commutative diagram

$$\begin{tikzcd}
\mathrm{H}^{p+q-1}(X(p-n+1,p)) \arrow[d, "\tilde{k}_1"'] \arrow[dr, bend left, "\partial"] & \\
\mathrm{H}^{p+q}(X(p)) \arrow[r, "j_1"] \arrow[d, "(i_1)^{n-1}"'] & \mathrm{H}^{p+q}(X(p,p+1)) \\
\mathrm{H}^{p+q}(X(p-n+1)), &
\end{tikzcd}$$

By Lemma 19.3 we have

$$B_n^{pq} = j_1(\mathrm{Ker}(i_1^{n-1})) = j_1(\mathrm{Im}(\tilde{k_1})) = \mathrm{Im}(\partial) = \mathrm{Ker}(\beta)$$

as desired. □

We will need the following general lemma:

Lemma 19.12. *Consider a commutative diagram*

$$\begin{tikzcd}
 & D \arrow[d, "h"] \arrow[dr, "k"] & \\
A \arrow[ur] \arrow[r, "f"'] & B \arrow[r, "g"'] & C
\end{tikzcd}$$

in an abelian category $\mathcal{A}$, where the bottom row is exact. Then g induces an isomorphism

$$g\colon \mathrm{Im}(h)/\mathrm{Im}(f) \xrightarrow{\sim} \mathrm{Im}(k).$$

Proof. Compute

$$\frac{\mathrm{Im}(h)}{\mathrm{Im}(f)} = \frac{\mathrm{Im}(h)}{\mathrm{Ker}(g)} \xrightarrow{\sim} \mathrm{Im}(gh) = \mathrm{Im}(k).$$ □

Proposition 19.13. *We have an isomorphism*

$$E_n^{pq} \cong \mathrm{Im}\big(\mathrm{H}^{p+q}(X(p,p+n)) \to \mathrm{H}^{p+q}(X(p-n+1,p+1))\big).$$

Proof. Consider the commutative diagram

$$\begin{tikzcd}
 & \mathrm{H}^{p+q}(X(p,p+n)) \arrow[d, "h"] \arrow[dr] & \\
\mathrm{H}^{p+q-1}(X(p-n+1,p)) \arrow[ur] \arrow[r, "f"'] & \mathrm{H}^{p+q}(X(p,p+1)) \arrow[r, "g"'] & \mathrm{H}^{p+q}(X(p-n+1,p+1)),
\end{tikzcd}$$

where the commutativity of the left triangle comes from the commutativity of (19.1) applied to

$$\begin{tikzcd}
X(p,p+n) \arrow[r] \arrow[d] & X(p-n+1,p+n) \arrow[r] \arrow[d] & X(p-n+1,p) \arrow[r] \arrow[d, equal] & X(p,p+n)[1] \arrow[d] \\
X(p,p+1) \arrow[r] & X(p-n+1,p+1) \arrow[r] & X(p-n+1,p) \arrow[r] & X(p,p+1)[1].
\end{tikzcd}$$

By Proposition 19.10 we have $Z_n^{pq} = \operatorname{Im}(h)$ and $B_n^{pq} = \operatorname{Im}(f)$, hence the claim follows from Lemma 19.12. □

Letting n go to infinity, we obtain the following consequence:

Corollary 19.14. *We have* $E_\infty^{pq} = \operatorname{Im}\big(\mathrm{H}^{p+q}(X(p)) \to \mathrm{H}^{p+q}(X(-\infty, p+1))\big)$.

Proof. Apply Proposition 19.13 and assumption $(*)$. □

Definition 19.15. Put

$$F^p\mathrm{H}^{p+q}(X(-\infty)) \coloneqq \operatorname{Im}\big(\mathrm{H}^{p+q}(X(p)) \to \mathrm{H}^{p+q}(X(-\infty))\big).$$

We summarize the previous discussion in the following theorem:

Theorem 19.16. *Let $\mathcal{A}$ be an abelian category and $X\colon \mathcal{I}^{\mathrm{op}} \to \mathsf{D}(\mathcal{A})$ a filtered object (see Definition 19.5. Suppose that the following condition is satisfied:*

- *For all fixed p, q and $n \gg 0$, the map $\mathrm{H}^{p+q}(X(p-n)) \xrightarrow{\sim} \mathrm{H}^{p+q}(X(-\infty))$ is an isomorphism and $\mathrm{H}^{p+q}(X(p+n)) = 0$.*

Then there is a converging spectral sequence

$$E_1^{pq} = \mathrm{H}^{p+q}(X(p,p+1)) \implies \mathrm{H}^{p+q}(X(-\infty)).$$

Proof. Following the discussion, it remains to prove that we have an isomorphism

$$E_\infty^{pq} \cong F^p\mathrm{H}^{p+q}(X(-\infty))/F^{p+1}\mathrm{H}^{p+q}(X(-\infty)).$$

To see this, apply Lemma 19.12 to the diagram

$$\begin{tikzcd}
& \mathrm{H}^{p+q}(X(p)) \arrow[d] \arrow[dr] & \\
\mathrm{H}^{p+q}(X(p+1)) \arrow[r] \arrow[ur] & \mathrm{H}^{p+q}(X(-\infty)) \arrow[r] & \mathrm{H}^{p+q}(X(-\infty,p+1)).
\end{tikzcd}$$

□

In the special case of a first quadrant spectral sequence we can extract more structure.

Construction 19.17 (edge maps)**.** Let

$$E_2^{pq} \implies H^{p+q}$$

be a convergent first quadrant spectral sequence, that is, $E_2^{pq} = 0$ provided $p < 0$ or $q < 0$.

Then the differentials into E_r^{0q} and out of E_r^{p0} are zero for all $r \geq 2$. In other words, we have inclusions

$$E_\infty^{0q} \subseteq \cdots \subseteq E_{r+1}^{0q} \subseteq E_r^{0q} \subseteq \cdots \subseteq E_2^{0q}$$

and quotient maps

$$E_2^{p0} \twoheadrightarrow \cdots \twoheadrightarrow E_r^{p0} \twoheadrightarrow E_{r+1}^{p0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p0}$$

Now, H^n has a filtration $0 = F^{n+1}H^n \subseteq F^nH^n \subseteq \cdots \subseteq F^1H^n \subseteq F^0H^n = H^n$ such that $E_\infty^{p,n-p} \cong F^pH^n/F^{p+1}H^n$. Hence, we obtain maps

$$H^q \twoheadrightarrow F^0H^q/F^1H^q \cong E_\infty^{0q} \hookrightarrow E_2^{0q}$$

and

$$E_2^{p0} \twoheadrightarrow E_\infty^{p0} \cong F^pH^p \subseteq H^p,$$

which are called the *edge maps* of the spectral sequence. In low degrees we obtain a *five-term exact sequence*

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \xrightarrow{d_2^{01}} E_2^{20} \to H^2.$$

In fact, there is even a *seven-term exact sequence*

$$0 \to E_2^{10} \to H^1 \to E_2^{01} \xrightarrow{d_2^{01}} E_2^{20} \to \operatorname{Ker}(H^2 \to E_2^{02}) \to E_2^{11} \xrightarrow{d_2^{11}} E_2^{30}.$$

§20. Examples of Spectral Sequences

Let us look at examples of spectral sequences in order to get a feeling for them.

Example 20.1. Let $\mathcal{A}$ be an abelian category. Let $X \xrightarrow{f} Y$ be a morphism in $\mathsf{D}(\mathcal{A})$, which gives rise to a distinguished triangle $X \to Y \to Z \to X[1]$. We view f as a filtered object by putting

$$X(p) = \begin{cases} 0, & \text{if } p \geq 2, \\ X, & \text{if } p = 1, \\ Y, & \text{if } p \leq 0. \end{cases}$$

The page $E_1^{pq} = \mathrm{H}^{p+q}(X(p,p+1))$ and E_2^{pq} are then given by

$$E_1^{pq}: \quad \begin{array}{c|l} q & \\ & \mathrm{H}^2(Z) \xrightarrow{d_2} \mathrm{H}^3(X) \\ & \mathrm{H}^1(Z) \xrightarrow{d_1} \mathrm{H}^2(X) \\ & \mathrm{H}^0(Z) \xrightarrow{d_0} \mathrm{H}^1(X) \\ \hline & \mathrm{H}^{-1}(Z) \xrightarrow{d_{-1}} \mathrm{H}^0(X) \quad p \end{array} \qquad E_2^{pq}: \quad \begin{array}{c|ll} q & & \\ & \mathrm{Ker}(d_2) & \mathrm{Coker}(d_2) \\ & \mathrm{Ker}(d_1) & \mathrm{Coker}(d_1) \\ & \mathrm{Ker}(d_0) & \mathrm{Coker}(d_0) \\ \hline & \mathrm{Ker}(d_{-1}) & \mathrm{Coker}(d_{-1}) \quad p \end{array}$$

Note that the spectral sequence is concentrated in the 0th and 1st column. The differentials of E_2^{pq} are all zero, hence $E_2^{pq} = E_\infty^{pq}$. The convergence means that we have short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & E_\infty^{1,n-1} \arrow[r] \arrow[d, equal] & \mathrm{H}^n(Y) \arrow[r] & E_\infty^{0,n} \arrow[r] \arrow[d, equal] & 0 \\ & \mathrm{Coker}(d_{n-1}) & & \mathrm{Ker}(d_n) & \end{tikzcd}$$

for $n \in \mathbb{Z}$. We can splice them together to recover the long exact sequence in cohomology:

$$\cdots \longrightarrow \mathrm{H}^{n-1}(Z) \xrightarrow{d_{n-1}} \mathrm{H}^n(X) \longrightarrow \mathrm{H}^n(Y) \longrightarrow \mathrm{H}^n(Z) \xrightarrow{d_n} \mathrm{H}^{n+1}(X) \longrightarrow \cdots$$

Observe that the spectral sequence allows us to compute $\mathrm{H}^n(Y)$ from the cohomologies $\mathrm{H}^n(X)$ and $\mathrm{H}^n(Z)$ only up to extensions!

20.1 The spectral sequence of a double complex

Definition 20.2. Let $\mathcal{A}$ be an additive category.

(i) A *double complex* consists of a tuple $(A^{\bullet,\bullet}, d_v^{\bullet,\bullet}, d_h^{\bullet,\bullet})$ consisting of objects $A^{i,j}$ and maps $d_v^{i,j}\colon A^{i,j} \to A^{i,j+1}$ and $d_h^{i,j}\colon A^{i,j} \to A^{i+1,j}$ such that $d_v^{i,j+1} \circ d_v^{i,j} = d_h^{i+1,j} \circ d_h^{i,j} = 0$ and

$d_v^{i+1,j} \circ d_h^{i,j} = d_h^{i,j+1} \circ d_v^{i,j}$. We depict the double complex as a lattice

$$\begin{tikzcd}
 & \vdots & \vdots & \vdots & \\
\cdots \arrow[r] & A^{i-1,j+1} \arrow[u] \arrow[r, "d_h^{i-1,j+1}"] & A^{i,j+1} \arrow[u] \arrow[r, "d_h^{i,j+1}"] & A^{i+1,j+1} \arrow[u] \arrow[r] & \cdots \\
\cdots \arrow[r] & A^{i-1,j} \arrow[u, "d_v^{i-1,j}"] \arrow[r, "d_h^{i-1,j}"] & A^{i,j} \arrow[u, "d_v^{i,j}"] \arrow[r, "d_h^{i,j}"] & A^{i+1,j} \arrow[u, "d_v^{i+1,j}"'] \arrow[r] & \cdots \\
\cdots \arrow[r] & A^{i-1,j-1} \arrow[u, "d_v^{i-1,j-1}"] \arrow[r, "d_h^{i-1,j-1}"'] & A^{i,j-1} \arrow[u, "d_v^{i,j-1}"] \arrow[r, "d_h^{i,j-1}"'] & A^{i+1,j-1} \arrow[u, "d_v^{i+1,j-1}"'] \arrow[r] & \cdots \\
 & \vdots \arrow[u] & \vdots \arrow[u] & \vdots \arrow[u] &
\end{tikzcd}$$

Note that a double complex is simply an object of $\mathsf{C}(\mathsf{C}(\mathcal{A}))$.

We call $A^{\bullet,\bullet}$ *bounded* if for all $n \in \mathbb{Z}$ there are only finitely many pairs (p, q) with $p + q = n$ and $A^{p,q} \neq 0$. For example, every first quadrant or third quadrant double complex is bounded.

(ii) Suppose that $\mathcal{A}$ admits infinite direct sums and let $A^{\bullet,\bullet}$ be a double complex. We denote by $(\mathrm{Tot}(A)^\bullet, d^\bullet)$ the complex defined by $\mathrm{Tot}(A)^n := \bigoplus_{\substack{p,q \\ p+q=n}} A^{p,q}$ with differentials $d^n \colon \mathrm{Tot}(A)^n \to \mathrm{Tot}(A)^{n+1}$ given by

$$d^n = \sum_{\substack{p,q \\ p+q=n}} d_h^{p,q} + (-1)^p d_v^{p,q}.$$

We call $\mathrm{Tot}(A)^\bullet$ the *total complex* of $A^{\bullet,\bullet}$.

Exercise 20.3. Let $\mathcal{A}$ be an additive category and $A^{\bullet,\bullet}$ a double complex. Let $B^{\bullet,\bullet}$ be the double complex given "transposing", *i.e.*, $B^{i,j} = A^{j,i}$, $d_{B,h}^{i,j} = d_{A,v}^{j,i}$ and $d_{B,v}^{i,j} = d_{A,h}^{j,i}$. Show that there is an isomorphism

$$\mathrm{Tot}(A^{\bullet,\bullet}) \xrightarrow{\sim} \mathrm{Tot}(B^{\bullet,\bullet})$$

in $\mathsf{C}(\mathcal{A})$.

Notation 20.4. Let $\mathcal{A}$ be an abelian category and $A^{\bullet,\bullet}$ a double complex in $\mathcal{A}$. For each fixed $p \in \mathbb{Z}$, we have a complex $(A^{p,\bullet}, d_v)$ in $\mathcal{A}$ with cohomology $\mathrm{H}_v^q(A^{p,\bullet}) \in \mathcal{A}$. These objects assemble into a complex

$$\big(\mathrm{H}_v^q(A^{\bullet,\bullet}), d_h\big) \in \mathsf{C}(\mathcal{A})$$

for each $q \in \mathbb{Z}$. We similarly denote by $\big(\mathrm{H}_h^p(A^{\bullet,\bullet}), d_v\big) \in \mathsf{C}(\mathcal{A})$ the complex assembled from the cohomologies $\mathrm{H}_h^p(A^{\bullet,q})$ of the complexes $(A^{\bullet,q}, d_h)$, for $q \in \mathbb{Z}$.

Proposition 20.5. *Let $\mathcal{A}$ be an abelian category. Let $A^{\bullet,\bullet}$ be a* bounded *double complex in $\mathcal{A}$. Then there exist two convergent spectral sequences*

$$ {}_{I}E_1^{p,q} = \mathrm{H}_v^q(A^{p,\bullet}) \implies \mathrm{H}^{p+q}(\mathrm{Tot}(A)) $$
$$ {}_{II}E_1^{p,q} = \mathrm{H}_h^q(A^{\bullet,p}) \implies \mathrm{H}^{p+q}(\mathrm{Tot}(A)), $$

where $d_1^{p,q} \colon {}_{I}E_1^{p,q} \to {}_{I}E_1^{p+1,q}$ is induced by d_h and $d_1^{p,q} \colon {}_{II}E_1^{p,q} \to {}_{II}E_1^{p+1,q}$ is induced by d_v.

Proof. Consider the “column filtration” given by $F^p\,\mathrm{Tot}(A) := \mathrm{Tot}(\sigma^{\geq p,\bullet}A) \subseteq \mathrm{Tot}(A)$. Concretely, we have $F^p\,\mathrm{Tot}(A)^n = \bigoplus_{\substack{i+j=n\\ i\geq p}} A^{i,j}$ as a subcomplex of $\mathrm{Tot}(A)$, which we may visualize as

$$\begin{array}{cc|cccc} 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ 0 & 0 & * & * & * & * \\ & & \geq p & & & \end{array}$$

Note that for fixed n we find p_0 such that $A^{p,n-p} = 0$ for all $p < p_0$ by our assumption that $A^{\bullet,\bullet}$ is bounded. Hence $\mathrm{H}^n(F^p\,\mathrm{Tot}(A)) \xrightarrow{\sim} \mathrm{H}^n(\mathrm{Tot}(A))$ is an isomorphism for all $p < p_0$. Moreover, it is clear that $\mathrm{H}^n(F^p\,\mathrm{Tot}(A)) = 0$ whenever $p > n$. We may thus apply Theorem 19.16 and obtain a convergent spectral sequence

$$ E_1^{pq} = \mathrm{H}^{p+q}(X(p,p+1)) \implies \mathrm{H}^{p+q}(\mathrm{Tot}(A)), $$

where X denotes the filtered object associated with $F^\bullet\,\mathrm{Tot}(A)$. Note that $X(p,p{+}1) = X(p)/X(p{+}1)$ identifies with the complex $(A^{p,\bullet-p}, (-1)^p d_v)$. We compute $E_1^{pq} = \mathrm{H}^{p+q}(X(p,p+1)) = \mathrm{H}_v^q(A^{p,\bullet})$, and the differential $d_1 \colon \mathrm{H}_v^q(A^{p,\bullet}) \to \mathrm{H}_v^q(A^{p+1,\bullet})$ is induced by d_h.

The second spectral sequence is obtained by applying the first result to the transposed double complex from Exercise 20.3, or by considering the “row filtration” $X(p) = \mathrm{Tot}(\sigma^{\bullet,\geq p}A)$ instead. The details are left to the reader. □

As an application, we can compute injective resolutions of (bounded below) complexes:

Exercise 20.6. Let $\mathcal{A}$ be an abelian category with enough injectives. Let

$$ A^\bullet = [\cdots \to 0 \to 0 \to A^0 \to A^1 \to A^2 \to \cdots] $$

be a complex concentrated in degrees ≥ 0.

(i) Construct a double complex $I^{\bullet,\bullet}$ such that $A^p \to I^{p,\bullet}$ is an injective resolution for all p. (Hint: split $A^\bullet$ into short exact sequences $0 \to Z^i \to A^i \to B^{i+1} \to 0$ and $0 \to B^i \to Z^i \to \mathrm{H}^i(A) \to 0$ for all $i \geq 0$, where $Z^i = \mathrm{Ker}(d^i)$ and $B^i = \mathrm{Im}(d^{i-1})$. Use injective resolutions of the B^i and $\mathrm{H}^i(A)$ and the Horseshoe lemma to construct injective resolutions of the Z^i and A^i.)

(ii) Show that the natural map $f \colon A^\bullet \to \mathrm{Tot}(I)^\bullet$ is a quasi-isomorphism. (Hint: Show first that the mapping cone of f is the total complex of the double complex $\widetilde{I}^{\bullet,\bullet}$ with $\widetilde{I}^{pq} = I^{pq}$ for all $p, q \geq 0$, $\widetilde{I}^{p,-1} = A^p$ for all p, and $\widetilde{I}^{pq} = 0$ otherwise.)

Exercise 20.7. Prove the snake lemma and the five lemma using spectral sequences.

20.2 The Grothendieck spectral sequence

We introduce some notation.

Notation 20.8. Let $\mathsf{D}(\mathcal{A})$ be the derived category of an abelian category $\mathcal{A}$.

(1) For any $n \in \mathbb{Z}$ we let $\mathsf{D}^{\leq n}(\mathcal{A}) \subseteq \mathsf{D}(\mathcal{A})$ be the full subcategory spanned by the objects X such that $\mathrm{H}^i(X) = 0$ for all $i > n$ (equivalently, $\tau^{\leq n} X \xrightarrow{\sim} X$).

We similarly define $\mathsf{D}^{\geq n}(\mathcal{A}) \subseteq \mathsf{D}(\mathcal{A})$ as the full subcategory spanned by the objects X such that $\mathrm{H}^i(X) = 0$ fo all $i < n$ (equivalently, $X \xrightarrow{\sim} \tau^{\geq n} X$). Observe that

$$\mathsf{D}^{-}(\mathcal{A}) = \bigcup_{n \in \mathbb{Z}} \mathsf{D}^{\leq n}(\mathcal{A}), \qquad \text{and} \qquad \mathsf{D}^{+}(\mathcal{A}) = \bigcup_{n \in \mathbb{Z}} \mathsf{D}^{\geq n}(\mathcal{A}).$$

(2) An exact functor $F \colon \mathsf{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{B})$ is called *left bounded* if there exists $n \in \mathbb{Z}$ such that $F(\mathsf{D}^{\geq 0}(\mathcal{A})) \subseteq \mathsf{D}^{\geq n}(\mathcal{B})$. If we can choose $n = 0$, then we call F *left t-exact*.

Similarly, F is called *right bounded* if there exists $n \in \mathbb{Z}$ such that $F(\mathsf{D}^{\leq 0}(\mathcal{A})) \subseteq \mathsf{D}^{\leq n}(\mathcal{B})$, and if we can choose $n = 0$, then we call F *right t-exact*.

F is called *bounded* if it is both left and right bounded, and we calld F *t-exact* if F is both left and right t-exact.

Proposition 20.9. *Let $\mathcal{A}, \mathcal{B}$ be abelian categories. Let $F \colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathcal{B})$ be an exact functor between derived categories and let $C \in \mathsf{C}(\mathcal{A})$. Suppose that one of the following conditions is satisfied:*

- *F is left bounded and $C \in \mathsf{C}^{+}(\mathcal{A})$;*
- *F is right bounded and $C \in \mathsf{C}^{-}(\mathcal{A})$;*
- *F is bounded;*
- *$C \in \mathsf{C}^{b}(\mathcal{A})$.*

Then there exist convergent spectral sequences

$$E_2^{pq} = (\mathrm{H}^p F)(\mathrm{H}^q(C)) \implies \mathrm{H}^{p+q}(F(C)) \qquad \textit{and}$$
$$E_1^{pq} = (\mathrm{H}^q F)(C^p) \implies \mathrm{H}^{p+q}(F(C)).$$

Here, $\mathrm{H}^p F \colon \mathcal{A} \to \mathcal{B}$ *denotes the composite* $\mathcal{A} \hookrightarrow \mathsf{D}(\mathcal{A}) \xrightarrow{F} \mathsf{D}(\mathcal{B}) \xrightarrow{\mathrm{H}^p} \mathcal{B}$.

Proof. Let us identify C with its image in $\mathsf{D}(\mathcal{A})$. Consider the filtration given by $F^p C \coloneqq \tau^{\leq -p} C$ from Example 19.7, and denote the associated filtered object by X. As F is an exact functor, the assignment $(FX)(p, q) \coloneqq F(X(p, q))$ defines a filtered object in $\mathsf{D}(\mathcal{B})$. We first claim that for fixed p, q and $n \gg 0$ the maps $\mathrm{H}^{p+q}(FX(p - n)) \xrightarrow{\sim} \mathrm{H}^{p+q}(FX(-\infty))$ are isomorphisms and $\mathrm{H}^{p+q}(FX(p + n)) = 0$.

- If $C \in \mathsf{D}^{\leq a}(\mathcal{A})$ for some $a \in \mathbb{Z}$, then we have $FX(p) = FC$ whenever $p + a \leq 0$, and hence $\mathrm{H}^{p+q}(FX(p - n)) = \mathrm{H}^{p+q}(FX(-\infty))$ for $n \geq p + a$.
- If $C \in \mathsf{D}^{\geq a}(\mathcal{A})$ for some $a \in \mathbb{Z}$, then $X(p + n) = \tau^{\leq -p-n} C = 0$ whenever $-n - p < a$.

- If F is left bounded, say $F(\mathsf{D}^{\geq 0}(\mathcal{A})) \subseteq \mathsf{D}^{\geq f_0}(\mathcal{B})$, then we have an exact sequence

$$\mathrm{H}^{p+q-1}(FX(-\infty, p-n)) \to \mathrm{H}^{p+q}(FX(p-n)) \to \mathrm{H}^{p+q}(FC) \to \mathrm{H}^{p+q}(FX(-\infty, p-n)).$$

  As $X(-\infty, p-n) = \tau^{>n-p}C$, we get $\mathrm{H}^{p+q-1}(FX(-\infty, p-n)) = 0 = \mathrm{H}^{p+q}(FX(-\infty, p-n))$, and therefore $\mathrm{H}^{p+q}(FX(p-n)) \xrightarrow{\sim} \mathrm{H}^{p+q}(FX(p-n-1))$ whenever $n - p + f_0 \geq p + q$.

- If F is right bounded, say $F(\mathsf{D}^{\leq 0}(\mathcal{A})) \subseteq \mathsf{D}^{\leq f_0}(\mathcal{B})$, then $\mathrm{H}^{p+q}(FX(p+n)) = 0$ whenever we have $f_0 - p - n < p + q$.

Thus, under our hypotheses, we may apply Theorem 19.16 to conclude that there is a convergent spectral sequence

$$E_1^{ij} = \mathrm{H}^{i+j}(FX(i, i+1)) \implies \mathrm{H}^{i+j}(F(C)).$$

We compute $\mathrm{H}^{i+j}(FX(i, i+1)) = \mathrm{H}^{i+j}(F(\mathrm{H}^{-i}(C)[i])) = \mathrm{H}^{2i+j}(F)(\mathrm{H}^{-i}(C))$. Now we make the substitution $(p, q) = (2i+j, -i)$, so that $p+q = i+j$. Under this new parametrization the differential has bidegree $(2, -1)$, hence our spectral sequence starts on the second page, *i.e.*, we get

$$E_2^{pq} = (\mathrm{H}^p F)(\mathrm{H}^q(C)) \implies \mathrm{H}^{p+q}(FC).$$

The same discussion applies *mutatis mutandis* if we consider the filtration $F^pC = \sigma^{\geq p}C$ instead. In this case we have $X(p, p+1) = C^p[-p]$ and hence we get a convergent spectral sequence

$$E_1^{pq} = \mathrm{H}^{p+q}(FC^p[-p]) = (\mathrm{H}^q F)(C^p) \implies \mathrm{H}^{p+q}(FC)$$

as desired. □

Theorem 20.10 (Grothendieck spectral sequence)**.** *Let $F\colon \mathcal{A} \to \mathcal{B}$ and $G\colon \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories. Suppose that $\mathcal{A}$ and $\mathcal{B}$ have enough injectives and that $F(I)$ is G-acyclic for all injective objects $I \in \mathcal{A}$.*

Then for all $A \in \mathcal{A}$ there is a convergent first quadrant spectral sequence

$$E_2^{pq} = \mathrm{R}^p G(\mathrm{R}^q F(A)) \implies \mathrm{R}^{p+q}(G \circ F)(A).$$

The edge maps are natural maps $\mathrm{R}^q(G \circ F)(A) \to G(\mathrm{R}^q F(A))$ and $\mathrm{R}^p G(F(A)) \to \mathrm{R}^p(G \circ F)(A)$ and we have a five-term exact sequence

$$0 \to \mathrm{R}^1 G(F(A)) \to \mathrm{R}^1(G \circ F)(A) \to G(\mathrm{R}^1 F(A)) \to \mathrm{R}^2 G(F(A)) \to \mathrm{R}^2(G \circ F)(A).$$

Proof. By Proposition 18.1 we have an isomorphism $\mathrm{R}(G \circ F) \xrightarrow{\sim} \mathrm{R}G \circ \mathrm{R}F$ of functors $\mathsf{D}^+(\mathcal{A}) \to \mathsf{D}(\mathcal{C})$. By Proposition 20.9 applied with $F = \mathrm{R}G$ and $C = \mathrm{R}F(A)$ we obtain the desired spectral sequence. The edge maps and the five-term exact sequence are given in Construction 19.17. □

Example 20.11 (Lyndon–Hochschild–Serre spectral sequence)**.** Let k be a field, G a finite group and $N \trianglelefteq G$ a normal subgroup. For every $V \in \mathsf{Rep}_k(G)$ we have $V^G = (V^N)^{G/N}$. Since $(\_)^N$ is right adjoint to the exact functor $\mathsf{Rep}_k(G/N) \to \mathsf{Rep}_k(G)$ given by inflation/restriction along $G \to G/N$, it follows that $(\_)^N$ preserves injective objects. Hence, in this case the Grothendieck spectral sequence yields

$$E_2^{pq} = \mathrm{H}^p(G/N, \mathrm{H}^q(N, V)) \implies \mathrm{H}^{p+q}(G, V),$$

where $\mathrm{H}^n(G, \_) = \mathrm{R}^n(\_)^G$ denotes group cohomology.

Example 20.12 (Leray spectral sequence)**.** Let $f\colon X \to Y$ be a continuous map between topological spaces. The direct image functor $f_*\colon \mathrm{Shv}(X, \mathsf{Ab}) \to \mathrm{Shv}(Y, \mathsf{Ab})$ sends injective sheaves to flabby sheaves, which are acyclic for the global sections functor $\Gamma(Y, -)$. Note that $\Gamma(Y, f_*(-)) = \Gamma(X, -)$. Hence, the Grothendieck spectral sequence yields

$$E_2^{pq} = \mathrm{H}^p(Y, \mathrm{R}^q f_*(\mathcal{F})) \implies \mathrm{H}^{p+q}(X, \mathcal{F})$$

for any sheaf $\mathcal{F} \in \mathrm{Shv}(X, \mathsf{Ab})$.

In other words, we can approximate the cohomology of $\mathcal{F}$ using the cohomologies of the higher direct images $\mathrm{R}^q f_*(\mathcal{F})$.

Exercise 20.13*.* Let $\mathcal{A}$ be an abelian category satisfying AB3 (infinite direct sums exist). Suppose that $\mathcal{A}$ has enough injective and enough projective objects. Show that the formation of infinite direct sums in $\mathcal{A}$ is exact. (Hint: A direct argument works (even without the assumption that $\mathcal{A}$ have enough projectives). For another argument consider the spectral sequence

$$E_2^{pq} = \mathrm{Ext}^p_{\mathcal{A}}\Big(\bigoplus\nolimits_i^{(q)} A_i, B\Big) \implies \prod_i \mathrm{Ext}^{p+q}_{\mathcal{A}}(A_i, B),$$

where $\bigoplus_i^{(q)}$ denotes the q-th left derived functor of $\bigoplus_i \colon \mathcal{A}^I \to \mathcal{A}$.)

Chapter 6

The Brown Representability Theorem

The goal in this section is to give a proof of the Brown representability theorem for triangulated catgeories. Its mean importance derives from the fact that it provides adjoint functor theorems. We roughly follow the treatment in [Kra07].

§21. Coherent Functors

Let $\mathcal{A}$ be an additive category. In this section we will describe a process of universally adjoining cokernels to $\mathcal{A}$. This construction will then be used to define the "abelian hull" of $\mathcal{A}$ (under a mild condition on $\mathcal{A}$).

Definition 21.1. We call $\mathsf{Mod}(\mathcal{A}) \coloneqq \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}^{\mathrm{op}}, \mathsf{Ab})$ the category of additive functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$.

Note that limits and colimits in $\mathsf{Mod}(\mathcal{A})$ exist and are computed pointwise. It follows that $\mathsf{Mod}(\mathcal{A})$ is an abelian category. The functor

$$\begin{aligned} \mathcal{Y} \colon \mathcal{A} &\to \mathsf{Mod}(\mathcal{A}), \\ A &\mapsto \mathrm{Hom}_{\mathcal{A}}(-, A) \end{aligned}$$

is additive and fully faithful by the Yoneda lemma.

We recall the following universal property of $\mathsf{Mod}(\mathcal{A})$, which informally states that $\mathsf{Mod}(\mathcal{A})$ is built from $\mathcal{A}$ by freely adjoining all colimits:

Theorem 21.2. *Let $\mathcal{A}, \mathcal{B}$ be additive categories, and suppose that $\mathcal{B}$ has all colimits. Then precomposition with $\mathcal{Y}$ induces an equivalence of categories*

$$\mathcal{Y}^* \colon \mathrm{Fun}^{\mathrm{colim}}(\mathsf{Mod}(\mathcal{A}), \mathcal{B}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}),$$

where $\mathrm{Fun}^{\mathrm{colim}}$ *denotes colimit preserving (additive) functors.*

Proof. The quasi-inverse is given by left Kan extension. Note first that, since $\mathcal{Y}$ is fully faithful, we obtain an adjunction

$$\mathcal{Y}_! \colon \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \rightleftarrows \mathrm{Fun}^{\mathrm{add}}(\mathsf{Mod}(\mathcal{A}), \mathcal{B}) : \mathcal{Y}^*.$$

We need to prove the following statements:

(a) $\mathcal{Y}_!$ is fully faithful, *i.e.*, the unit $\mathrm{id} \xrightarrow{\sim} \mathcal{Y}^*\mathcal{Y}_!$ is an isomorphism.

(b) $\mathcal{Y}_!(F)\colon \mathsf{Mod}(\mathcal{A}) \to \mathcal{B}$ preserves all colimits, for $F \in \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B})$.

(c) The restriction of $\mathcal{Y}^*$ to $\mathrm{Fun}^{\mathrm{colim}}(\mathsf{Mod}(\mathcal{A}), \mathcal{B})$ is conservative.

Let us assume these statements and show how to prove the statement: By (b) we obtain an adjunction $\mathcal{Y}_!\colon \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \rightleftarrows \mathrm{Fun}^{\mathrm{colim}}(\mathsf{Mod}(\mathcal{A}), \mathcal{B}) : \mathcal{Y}^*$. By (a), the unit $\eta\colon \mathrm{id} \xrightarrow{\sim} \mathcal{Y}^*\mathcal{Y}_!$ is an isomorphism. It remains to show that the counit $\varepsilon\colon \mathcal{Y}_!\mathcal{Y}^* \to \mathrm{id}$ is an isomorphism. But note that the triangle identity says that the composite

$$\mathcal{Y}^* \xrightarrow{\eta\mathcal{Y}^*} \mathcal{Y}^*\mathcal{Y}_!\mathcal{Y}^* \xrightarrow{\mathcal{Y}^*\varepsilon} \mathcal{Y}^*$$

is the identity. It follows that $\mathcal{Y}^*\varepsilon$ is an isomorphism. As $\mathcal{Y}^*$ is conservative by (c), we deduce that ε is an isomorphism.

Part (a) follows from Corollary 13.16.

We now prove (b). Let $F \in \mathrm{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B})$ and consider the functor

$$\begin{aligned} \mathrm{Sing}_F\colon \mathcal{B} &\to \mathsf{Mod}(\mathcal{A}), \\ B &\mapsto \mathrm{Hom}_{\mathcal{B}}(F(-), B). \end{aligned}$$

We claim that Sing_F is a right adjoint of $\mathcal{Y}_!F$. For any $M \in \mathsf{Mod}(\mathcal{A})$ and $B \in \mathcal{B}$ we have natural isomorphisms

$$\begin{aligned} \mathrm{Hom}(M, \mathrm{Sing}_F(B)) &\cong \mathrm{Hom}\big(\mathrm{Hom}(\mathcal{Y}(-), M), \mathrm{Hom}_{\mathcal{B}}(F(-), B)\big) && \text{(Yoneda lemma)} \\ &\cong \mathrm{Hom}\big(\mathrm{Hom}(-, M), \mathcal{Y}_* \mathrm{Hom}_{\mathcal{B}}(F(-), B)\big) && \text{(right Kan extension)} \\ &\cong \mathrm{Hom}\big(\mathrm{Hom}(-, M), \mathrm{Hom}_{\mathcal{B}}(\mathcal{Y}_!F(-), B)\big) \\ &\cong \mathrm{Hom}_{\mathcal{B}}\big((\mathcal{Y}_!F)(M), B\big). \end{aligned}$$

The penultimate isomorphism holds, because $\mathcal{Y}_!F$ is a pointwise Kan extension by Proposition 13.14.

It remains to prove (c). Let $\alpha\colon F \to G$ be a natural transformation of colimit-preserving functors $F, G\colon \mathsf{Mod}(\mathcal{A}) \to \mathcal{B}$ such that $\mathcal{Y}^*(\alpha)$ is an isomorphism. Let $M \in \mathsf{Mod}(\mathcal{A})$. By Lemma 13.13 the canonical map

$$\operatorname*{colim}_{(A,f)\in\mathcal{Y}/M} \mathcal{Y}(A) \xrightarrow{\sim} M$$

is an isomorphism. Since F, G commute with colimits, the diagram

$$\begin{tikzcd} F(M) \arrow[r, "\sim"] \arrow[d, "\alpha_M"'] & \mathrm{colim}_{(A,f)\in\mathcal{Y}/M} \mathcal{Y}^*(F)(A) \arrow[d, "\mathrm{colim}\,\mathcal{Y}^*(\alpha)_A", "\sim"'] \\ G(M) \arrow[r, "\sim"] & \mathrm{colim}_{(A,f)\in\mathcal{Y}/M} \mathcal{Y}^*(G)(A) \end{tikzcd}$$

is commutative. We deduce that α_M is an isomorphism as desired. □

Definition 21.3. Let $\mathcal{A}$ be an additive category. A functor $F\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ is called *coherent* if there exist $A, B \in \mathcal{A}$ and an exact sequence

$$\mathrm{Hom}_{\mathcal{A}}(-, B) \to \mathrm{Hom}_{\mathcal{A}}(-, A) \to F \to 0$$

in $\mathsf{Mod}(\mathcal{A})$. We denote by $\mathrm{Coh}(\mathcal{A}) \subseteq \mathsf{Mod}(\mathcal{A})$ the full subcategory of coherent functors.

Definition 21.4. Let $\mathcal{A}$ be an additive category. We say that $\mathcal{A}$ has *weak kernels* if for all maps $A \to B$ in $\mathcal{A}$ there exists a map $C \to A$ such that the sequence

$$\operatorname{Hom}_{\mathcal{A}}(-, C) \to \operatorname{Hom}_{\mathcal{A}}(-, A) \to \operatorname{Hom}_{\mathcal{A}}(-, B)$$

in $\mathsf{Mod}(\mathcal{A})$ is exact.

Example 21.5. Every triangulated category has weak kernels by Proposition 4.7(ii).

The main result about coherent functors is the following:

Proposition 21.6. *Let $\mathcal{A}$ be an additive category.*

(i) The Yoneda embedding $\mathcal{Y}\colon \mathcal{A} \to \mathsf{Mod}(\mathcal{A})$ factors through an embedding $\mathcal{Y}_c\colon \mathcal{A} \to \operatorname{Coh}(\mathcal{A})$.

(ii) $\operatorname{Coh}(\mathcal{A})$ is an additive category and is closed under cokernels in $\mathsf{Mod}(\mathcal{A})$. Moreover each $\operatorname{Hom}_{\mathcal{A}}(-, A)$ is projective and $\operatorname{Coh}(\mathcal{A})$ has enough projective objects.

(iii) Coherent functors $\mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ preserve all products (which exist in $\mathcal{A}^{\mathrm{op}}$).

(iv) If $\mathcal{A}$ has all direct sums, then $\operatorname{Coh}(\mathcal{A})$ has all direct sums and $\mathcal{Y}_c$ preserves them.

(v) The category $\operatorname{Coh}(\mathcal{A})$ has the following universal property: Let $\mathcal{B}$ be an additive category in which cokernels exist. Then precomposition with $\mathcal{Y}_c$ induces an equivalence of categories

$$\mathcal{Y}_c^*\colon \operatorname{Fun}^{\mathrm{rex}}(\operatorname{Coh}(\mathcal{A}), \mathcal{B}) \xrightarrow{\sim} \operatorname{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}),$$

where $\operatorname{Fun}^{\mathrm{rex}}$ denotes additive functors which preserve cokernels.

(vi) Suppose that $\mathcal{A}$ has weak kernels. Then $\operatorname{Coh}(\mathcal{A}) \subseteq \mathsf{Mod}(\mathcal{A})$ is an abelian subcategory.

Proof. Part (i) is obvious.

Let us prove (ii). It is clear that $\operatorname{Coh}(\mathcal{A})$ is additive and that each $\operatorname{Hom}_{\mathcal{A}}(-, A)$ is projective (by the Yoneda lemma), so that $\operatorname{Coh}(\mathcal{A})$ has enough projectives. It remains to prove that $\operatorname{Coh}(\mathcal{A})$ is closed under cokernels in $\mathsf{Mod}(\mathcal{A})$. Let $\varphi\colon F \to F'$ be a map in $\operatorname{Coh}(\mathcal{A})$ with cokernel $F' \to F''$ in $\mathsf{Mod}(\mathcal{A})$. Choose presentations $\mathcal{Y}(B) \to \mathcal{Y}(A) \to F \to 0$ and $\mathcal{Y}(B') \to \mathcal{Y}(A) \to F' \to 0$ and consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}(-, B) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, B') \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, A \oplus B') \arrow[d, "g"] & \\
\operatorname{Hom}_{\mathcal{A}}(-, A) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, A') \arrow[r, equal] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(-, A') \arrow[d, "f"] & \\
F \arrow[r] & F' \arrow[r] & F'' \arrow[r] & 0;
\end{tikzcd}$$

the horizontal maps on the left can be chosen by the projectivity of $\operatorname{Hom}_{\mathcal{A}}(-, A)$ and $\operatorname{Hom}_{\mathcal{A}}(-, B)$. The map f is obviously surjective, and a diagram chase shows that the image of g coincides with the kernel of f. Hence F'' is coherent and a cokernel in $\operatorname{Coh}(\mathcal{A})$.

We now prove (iii). Let F be a coherent functor with presentation $\mathcal{Y}(B) \to \mathcal{Y}(A) \to F \to 0$. Let $(X_i)_{i \in I}$ be a family of objects in in $\mathcal{A}$ such that the direct sum $\bigoplus_{i \in I} X_i$ exists. Then we have a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_{\mathcal{A}}\Big(\bigoplus_{i\in I} X_i, B\Big) \arrow[r] \arrow[d, "\sim"] & \operatorname{Hom}_{\mathcal{A}}\Big(\bigoplus_{i\in I} X_i, A\Big) \arrow[r] \arrow[d, "\sim"] & F\Big(\bigoplus_{i\in I} X_i\Big) \arrow[r] \arrow[d] & 0 \\
\prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(X_i, B) \arrow[r] & \prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(X_i, A) \arrow[r] & \prod_{i\in I} F(X_i) \arrow[r] & 0,
\end{tikzcd}$$

so we conclude from the five lemma that $F\colon \mathcal{A}^{\mathrm{op}} \to \mathsf{Ab}$ preserves all products (which exist in $\mathcal{A}^{\mathrm{op}}$).

For part (iv), assume that $\mathcal{A}$ has all direct sums and consider the commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}\big(\mathcal{Y}_c\Big(\bigoplus_{i\in I} X_i\Big), F\big) \arrow[r, "\sim"] \arrow[d] & F\Big(\bigoplus_{i\in I} X_i\Big) \arrow[d, "\sim"] \\
\prod_{i\in I} \operatorname{Hom}\big(\mathcal{Y}_c(X_i), F\big) \arrow[r, "\sim"] & \prod_{i\in I} F(X_i).
\end{tikzcd}$$

We deduce that the left vertical map is an isomorphism, which shows that $\mathcal{Y}_c$ preserves all direct sums. Let now $(F_i)_{i\in I}$ be a family in $\operatorname{Coh}(\mathcal{A})$ with presentations $\mathcal{Y}(B_i) \to \mathcal{Y}(A_i) \to F_i \to 0$. Put $F \coloneqq \operatorname{Coker}(\mathcal{Y}(\bigoplus_{i\in I} B_i) \to \mathcal{Y}(\bigoplus_{i\in I} A_i))$. For all $G \in \operatorname{Coh}(\mathcal{A})$ we then have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{Hom}(F, G) \arrow[r] \arrow[d] & \operatorname{Hom}(\mathcal{Y}_c\Big(\bigoplus_{i\in I} A_i\Big), G) \arrow[r] \arrow[d, "\sim"] & \operatorname{Hom}(\mathcal{Y}_c\Big(\bigoplus_{i\in I} B_i\Big), G) \arrow[d, "\sim"] \\
0 \arrow[r] & \prod_{i\in I} \operatorname{Hom}(F_i, G) \arrow[r] & \prod_{i\in I} \operatorname{Hom}(\mathcal{Y}_c(A_i), G) \arrow[r] & \prod_{i\in I} \operatorname{Hom}(\mathcal{Y}_c(B_i), G).
\end{tikzcd}$$

The five lemma shows that the left vertical map is an isomorphism, *i.e.*, F is the direct sum of the F_i in $\operatorname{Coh}(\mathcal{A})$. Note here that the direct sum is *not* formed pointwise!

We now prove (v). Choose an embedding $j\colon \mathcal{B} \hookrightarrow \mathcal{B}'$ into a cocomplete category $\mathcal{B}'$ such that j preserves cokernels. For example $\mathcal{B}' = \operatorname{Fun}^{\mathrm{add}}(\mathcal{B}, \mathsf{Ab}^{\mathrm{op}})$ has the desired properties, where j is the opposite Yoneda embedding given by $j(B) = \operatorname{Hom}_{\mathcal{B}}(-, B)^{\mathrm{op}}\colon \mathcal{B} \to \mathsf{Ab}^{\mathrm{op}}$. We have a commutative diagram

$$\begin{tikzcd}
\operatorname{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}) \arrow[d, hook] & \operatorname{Fun}^{\mathrm{add}}(\operatorname{Coh}(\mathcal{A}), \mathcal{B}) \arrow[l, "\mathcal{Y}_c^*"'] \arrow[d, hook] \\
\operatorname{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}') & \operatorname{Fun}^{\mathrm{add}}(\operatorname{Coh}(\mathcal{A}), \mathcal{B}'), \arrow[l, "\mathcal{Y}_c^*"]
\end{tikzcd}$$

where the vertical maps are fully faithful since j is fully faithful. Moreover, we have an adjunction

$$\mathcal{Y}_{c!}\colon \operatorname{Fun}^{\mathrm{add}}(\mathcal{A}, \mathcal{B}') \rightleftarrows \operatorname{Fun}^{\mathrm{add}}(\operatorname{Coh}(\mathcal{A}), \mathcal{B}') :\mathcal{Y}_c^*.$$

We first show that the essential image of $\mathcal{Y}_{c!}$ lies in $\operatorname{Fun}^{\mathrm{rex}}(\operatorname{Coh}(\mathcal{A}), \mathcal{B}')$. We denote by $i\colon \operatorname{Coh}(\mathcal{A}) \hookrightarrow \mathsf{Mod}(\mathcal{A})$ the inclusion, which commutes with cokernels by (ii). Again by Corollary 13.16 we know that the unit $\mathrm{id} \xrightarrow{\sim} i^* i_!$ is an isomorphism. Now, let $F\colon \mathcal{A} \to \mathcal{B}'$ be an additive functor. Then $\mathcal{Y}_! F\colon \mathsf{Mod}(\mathcal{A}) \to \mathcal{B}'$ preserves all colimits, hence its restriction $i^* \mathcal{Y}_! F = i^* i_! \mathcal{Y}_{c!} F \cong \mathcal{Y}_{c!} F$ preserves cokernels as desired.

As every coherent functor is a cokernel of representables and $\mathcal{B}$ is admits cokernels which are preserved by j, it follows that $\mathcal{Y}_{c!}$ restricts to a functor $\mathrm{Fun}^{\mathrm{add}}(\mathcal{A},\mathcal{B}) \to \mathrm{Fun}^{\mathrm{rex}}(\mathrm{Coh}(\mathcal{A}),\mathcal{B})$. Hence, we obtain an adjunction

$$\mathcal{Y}_{c!}\colon \mathrm{Fun}^{\mathrm{add}}(\mathcal{A},\mathcal{B}) \rightleftarrows \mathrm{Fun}^{\mathrm{rex}}(\mathrm{Coh}(\mathcal{A}),\mathcal{B}) : \mathcal{Y}_c^*.$$

Again, $\mathcal{Y}_{c!}$ is fully faithful by Corollary 13.16, and $\mathcal{Y}_c^*$ is conservative by the five lemma. As in the proof of Theorem 21.2 we deduce that $\mathcal{Y}_{c!}$ is an equivalence of categories with quasi-inverse $\mathcal{Y}_c^*$.

We finally prove (vi). We will show that $\mathrm{Coh}(\mathcal{A}) \subseteq \mathsf{Mod}(\mathcal{A})$ is an abelian subcategory. By (ii), $\mathrm{Coh}(\mathcal{A})$ is closed under cokernels. It remains to prove that $\mathrm{Coh}(\mathcal{A})$ is closed under kernels. So let $0 \to F' \to F \to F''$ be an exact sequence in $\mathsf{Mod}(\mathcal{A})$ with $F, F'' \in \mathrm{Coh}(\mathcal{A})$. We will prove that F' is coherent. Let $\mathrm{Hom}_{\mathcal{A}}(-,B) \to \mathrm{Hom}_{\mathcal{A}}(-,A) \to F \to 0$ and $\mathrm{Hom}_{\mathcal{A}}(-,B'') \to \mathrm{Hom}_{\mathcal{A}}(-,A'') \to F'' \to 0$ be presentations of F and F'', respectively, and consider the following commutative diagram:

$$\begin{tikzcd}
 & & \mathrm{Hom}_{\mathcal{A}}(-,B) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(-,B'') \arrow[d] & \\
 & & \mathrm{Hom}_{\mathcal{A}}(-,A) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathcal{A}}(-,A'') \arrow[d] & \\
0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0,
\end{tikzcd}$$

where the horizontal maps can be chosen, because $\mathrm{Hom}_{\mathcal{A}}(-,A)$ and $\mathrm{Hom}_{\mathcal{A}}(-,B)$ are projective by (ii). Since $\mathcal{A}$ has weak kernels, we find $A' \in \mathcal{A}$ and a map $A' \to A \oplus B''$ such that the sequence $\mathrm{Hom}_{\mathcal{A}}(-,A') \to \mathrm{Hom}_{\mathcal{A}}(-,A \oplus B'') \to \mathrm{Hom}_{\mathcal{A}}(-,A'')$ is exact. Similarly, we find $B' \in \mathcal{A}$ and a map $B' \to A' \oplus B$ such that the sequence $\mathrm{Hom}_{\mathcal{A}}(-,B') \to \mathrm{Hom}_{\mathcal{A}}(-,A' \oplus B) \to \mathrm{Hom}_{\mathcal{A}}(-,A)$ is exact. A diagram chase then shows that

$$\mathrm{Hom}_{\mathcal{A}}(-,B') \to \mathrm{Hom}_{\mathcal{A}}(-,A') \to F' \to 0$$

is exact, *i.e.*, F' is coherent. □

Exercise 21.7. Let $\mathcal{A}$ be an abelian category and suppose that $\mathcal{A}$ has enough projective objects. Let $\mathcal{P} \subseteq \mathcal{A}$ be the full subcategory of projective objects. Show that the functor

$$\begin{aligned} \mathcal{A} &\xrightarrow{\sim} \mathrm{Coh}(\mathcal{P}), \\ A &\mapsto \mathrm{Hom}_{\mathcal{A}}(-,A)\big|_{\mathcal{P}} \end{aligned}$$

is an equivalence of categories.

§22. The Abelianization of a Triangulated Category

Proposition 22.1. *Let $(\mathcal{C}, T)$ be a triangulated category.*

(i) The category $\mathrm{Coh}(\mathcal{C})$ is abelian and the Yoneda functor $\mathcal{Y}_c\colon \mathcal{C} \to \mathrm{Coh}(\mathcal{C})$ is cohomological.

(ii) Let $\mathcal{A}$ be an abelian category. Then precomposition with $\mathcal{Y}_c$ induces an equivalence

$$\mathcal{Y}_c^*\colon \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Coh}(\mathcal{C}),\mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C},\mathcal{A})$$

of categories, where $\mathrm{Fun}^{\mathrm{coh}}$ is the category of cohomological functors and $\mathrm{Fun}^{\mathrm{ex}}$ denotes exact functors (between abelian categories).

Proof. A weak kernel of a map $g\colon Y \to Z$ in $\mathcal{C}$ is a map $f\colon X \to Y$ sitting in a distinguished triangle $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} T(X)$, because by Proposition 4.7(ii) the sequence

$$\cdots \to \mathrm{Hom}_{\mathcal{C}}(C, X) \to \mathrm{Hom}_{\mathcal{C}}(C, Y) \to \mathrm{Hom}_{\mathcal{C}}(C, Z) \to \mathrm{Hom}_{\mathcal{C}}(C, T(X)) \to \cdots$$

is exact for every $C \in \mathcal{C}$. Hence, by Proposition 21.6(vi) $\mathrm{Coh}(\mathcal{C})$ is abelian, and clearly $\mathcal{Y}_c\colon \mathcal{C} \to \mathrm{Coh}(\mathcal{C})$ is cohomological.

It remains to prove (ii). By Proposition 21.6(v) we have an equivalence of categories

$$\mathcal{Y}_c^*\colon\ \mathrm{Fun}^{\mathrm{rex}}(\mathrm{Coh}(\mathcal{C}), \mathcal{A}) \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{add}}(\mathcal{C}, \mathcal{A}).$$

Since $\mathcal{Y}_c\colon \mathcal{C} \to \mathrm{Coh}(\mathcal{C})$ is cohomological, it is clear that for any exact functor $\overline{H}\colon\ \mathrm{Coh}(\mathcal{C}) \to \mathcal{A}$, the composed functor $\overline{H} \circ \mathcal{Y}_c$ is cohomological. Conversely, let $H\colon \mathcal{C} \to \mathcal{A}$ be a cohomological functor, and let $\overline{H}\colon\ \mathrm{Coh}(\mathcal{C}) \to \mathcal{A}$ be a right exact functor such that $\overline{H} \circ \mathcal{Y}_c \cong H$. We need to show that $\overline{H}$ is also left exact. To this end, let $0 \to F' \to F \to F'' \to 0$ be a short exact sequence in $\mathrm{Coh}(\mathcal{C})$. We may then construct a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{Y}_c(C') \arrow[r] \arrow[d] & \mathcal{Y}_c(C) \arrow[r] \arrow[d] & \mathcal{Y}_c(C'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Y}_c(B') \arrow[r] \arrow[d] & \mathcal{Y}_c(B) \arrow[r] \arrow[d] & \mathcal{Y}_c(B'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{Y}_c(A') \arrow[r] \arrow[d, two heads] & \mathcal{Y}_c(A) \arrow[r] \arrow[d, two heads] & \mathcal{Y}_c(A'') \arrow[r] \arrow[d, two heads] & 0 \\
0 \arrow[r] & F' \arrow[r] & F \arrow[r] & F'' \arrow[r] & 0,
\end{tikzcd}$$

where the first three rows are split exact (as every $\mathcal{Y}_c(X)$ is projective), and where $C'' \xrightarrow{f} B'' \to A'' \to T(C'')$ is a distinguished triangle in $\mathcal{C}$. Applying $\overline{H}$ and using $\overline{H} \circ \mathcal{Y}_c \cong H$, we obtain a commutative diagram

$$\begin{tikzcd}
 & & H(C) \arrow[r] \arrow[d] & H(C'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H(B') \arrow[r] \arrow[d] & H(B) \arrow[r] \arrow[d] & H(B'') \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H(A') \arrow[r] \arrow[d] & H(A) \arrow[r] \arrow[d] & H(A'') \arrow[r] \arrow[d] & 0 \\
 & \overline{H}(F') \arrow[r] & \overline{H}(F) \arrow[r] & \overline{H}(F'') \arrow[r] & 0
\end{tikzcd}$$

where the rows and the first and third columns are exact, and where the second column is a complex which is exact at $H(A)$. A diagram chase now shows that $\overline{H}(F') \to \overline{H}(F)$ is injective, hence $\overline{H}$ is exact. $\square$

Proposition 22.2. *Let $(\mathcal{C}, T)$ be a triangulated category. Then there is a canonical equivalence*

$$\mathrm{Coh}(\mathcal{C})^{\mathrm{op}} \cong \mathrm{Coh}(\mathcal{C}^{\mathrm{op}})$$

of abelian categories.

Proof. It suffices to check that the cohomological functor $\mathcal{Y}_c^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Coh}(\mathcal{C})^{\mathrm{op}}$ satisfies the universal property of $\mathrm{Coh}(\mathcal{C}^{\mathrm{op}})$ (see Proposition 22.1). Let $\mathcal{A}$ be an abelian category. Then we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Fun}^{\mathrm{ex}}(\mathrm{Coh}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}) \arrow[r, equal] \arrow[d, "(\mathcal{Y}_c^{\mathrm{op}})^*"'] & \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Coh}(\mathcal{C}), \mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \arrow[d, "\simeq"', "(\mathcal{Y}_c^*)^{\mathrm{op}}"] \\
\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \arrow[r, equal] & \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{A}^{\mathrm{op}})^{\mathrm{op}}.
\end{tikzcd}$$

We deduce that the left vertical map is an isomorphism, which proves the claim. □

Corollary 22.3. *Let $(\mathcal{C}, T)$ be a triangulated category. Then $\mathrm{Coh}(\mathcal{C})$ is an abelian Frobenius category.*

Proof. Note that $\mathrm{Coh}(\mathcal{C})$ is abelian by Proposition 22.1. We already know by Proposition 21.6(ii) that the category $\mathrm{Coh}(\mathcal{C})$ has enough projectives, and that the representable functors $\mathcal{Y}_c(X) \in \mathrm{Coh}(\mathcal{C})$ are projective for each $X \in \mathcal{C}$. In order to show that $\mathrm{Coh}(\mathcal{C})$ is Frobenius (*i.e.*, that it also has enough injectives and that the classes of projective and injective objects coincide), it suffices to show that there is an equivalence

$$\mathrm{Coh}(\mathcal{C})^{\mathrm{op}} \cong \mathrm{Coh}(\mathcal{C}^{\mathrm{op}})$$

of categories. To this end, it suffices to check that $\mathrm{Coh}(\mathcal{C})^{\mathrm{op}}$ satisfies the universal property of $\mathrm{Coh}(\mathcal{C}^{\mathrm{op}})$: The map $\mathcal{Y}_c^{\mathrm{op}} \colon \mathcal{C}^{\mathrm{op}} \to \mathrm{Coh}(\mathcal{C})^{\mathrm{op}}$ given by the opposite of the Yoneda embedding $\mathcal{C} \to \mathrm{Coh}(\mathcal{C})$ is clearly cohomological. Let now $\mathcal{A}$ be an abelian category. Then we have equivalences

$$\mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}^{\mathrm{op}}, \mathcal{A}) \cong \mathrm{Fun}^{\mathrm{coh}}(\mathcal{C}, \mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \xrightarrow{\sim} \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Coh}(\mathcal{C}), \mathcal{A}^{\mathrm{op}})^{\mathrm{op}} \cong \mathrm{Fun}^{\mathrm{ex}}(\mathrm{Coh}(\mathcal{C})^{\mathrm{op}}, \mathcal{A}),$$

and it is easily checked that it is induced by precomposition with $\mathcal{Y}_c^{\mathrm{op}}$. This finishes the proof. □

Corollary 22.4. *Let $(\mathcal{C}, T)$ be a triangulated category with arbitrary direct sums. Then $\mathrm{Coh}(\mathcal{C})$ is AB4 (the formation of arbitrary direct sums is exact).*

Proof. $\mathrm{Coh}(\mathcal{C})$ has all direct sums by Proposition 21.6(iv) and enough injectives by Corollary 22.3. Hence $\mathrm{Coh}(\mathcal{C})$ is AB4 (cf. Exercise 20.13). □

§23. Brown Representability

Definition 23.1. Let $(\mathcal{C}, T)$ be a triangulated category admitting countable direct sums. A set $\mathcal{P}_0$ of objects in $\mathcal{C}$ is called a set of *generators* if it satisfies

(PG1) If $X \in \mathcal{C}$ is such that $\mathrm{Hom}_{\mathcal{C}}(T^n(P), X) = 0$ for all $n \in \mathbb{Z}$ and $P \in \mathcal{P}_0$, then $X = 0$.

Moreover, we call $\mathcal{P}_0$ a set of *perfect generators* if in addition it satisfies:

(PG2) Given a countable family of maps $\{X_i \xrightarrow{f_i} Y_i\}_{i\in I}$ in $\mathcal{C}$ such that $\operatorname{Hom}_{\mathcal{C}}(P,X_i) \twoheadrightarrow \operatorname{Hom}_{\mathcal{C}}(P,Y_i)$ is surjective for all i and $P \in \mathcal{P}_0$, then the induced map

$$\operatorname{Hom}_{\mathcal{C}}\Big(P,\bigoplus_{i\in I} X_i\Big) \twoheadrightarrow \operatorname{Hom}_{\mathcal{C}}\Big(P,\bigoplus_{i\in I} Y_i\Big)$$

is surjective for all $P \in \mathcal{P}_0$.

Lemma 23.2. *Let $\mathcal{C}$ be an additive category with arbitrary direct sums and weak kernels. Let $\mathcal{P}_0$ be a set of objects in $\mathcal{C}$ and denote by $\mathcal{P} \subseteq \mathcal{C}$ the full subcategory spanned by all direct sums of copies of objects in $\mathcal{P}_0$.*

(i) The category $\mathcal{P}$ has weak kernels and $\operatorname{Coh}(\mathcal{P})$ *is abelian.*

(ii) The functor $\operatorname{Coh}(\mathcal{C}) \to \operatorname{Coh}(\mathcal{P})$, $F \mapsto F\big|_{\mathcal{P}}$ *is exact.*

(iii) The composite $\mathcal{C} \xrightarrow{\mathcal{Y}_c} \operatorname{Coh}(\mathcal{C}) \to \operatorname{Coh}(\mathcal{P})$ *preserves countable direct sums if and only if (PG2) holds.*

Proof. We first observe that every $X \in \mathcal{C}$ has an approximation in $\mathcal{P}$, that is, there exists a map $P \to X$ with $P \in \mathcal{P}$ such that $\operatorname{Hom}(Q,P) \twoheadrightarrow \operatorname{Hom}(Q,X)$ is surjective, for all $Q \in \mathcal{P}$; just consider $P = \bigoplus_{Q\in\mathcal{P}_0} Q^{\oplus \operatorname{Hom}(Q,X)}$ and the canonical map $P \to X$.

For part (i) it suffices to show that $\mathcal{P}$ has weak kernels as then Proposition 21.6(vi) shows that $\operatorname{Coh}(\mathcal{P})$ is abelian. Let $f\colon P_1 \to P_2$ be a map in $\mathcal{P}$. Then the composite of a weak kernel $X \to P_1$ in $\mathcal{C}$ and an approximation $P_0 \to X$ is a weak kernel for f.

For part (ii) the only non-trivial statement is that for $F \in \operatorname{Coh}(\mathcal{C})$ the restriction $F\big|_{\mathcal{P}}$ is again coherent. We may assume $F = \operatorname{Hom}_{\mathcal{C}}(-,X)$ for some $X \in \mathcal{C}$. Consider the sequence $P' \xrightarrow{f} X' \xrightarrow{g} P \xrightarrow{h} X$, where h and f are approximations and g is a weak kernel of h in $\mathcal{C}$. By construction the sequence

$$\operatorname{Hom}_{\mathcal{P}}(-,P') \to \operatorname{Hom}_{\mathcal{P}}(-,P) \to \operatorname{Hom}_{\mathcal{C}}(-,X)\big|_{\mathcal{P}} \to 0$$

is exact.

Finally, we need to show (iii). Denote by $\varphi\colon \mathcal{P} \hookrightarrow \mathcal{C}$ the inclusion and by $\varphi^*\colon \operatorname{Coh}(\mathcal{C}) \to \operatorname{Coh}(\mathcal{P})$ the induced restriction functor.

Step 1: $\varphi^*\mathcal{Y}_c$ commutes with countable direct sums if and only if so does φ^*.

The "if"-direction is clear since $\mathcal{Y}_c$ commutes with countable direct sums. Suppose now that $\varphi^*\mathcal{Y}_c$ commutes with countable direct sums. Let $\{F_i\}_{i\in I}$ be a countable family in $\operatorname{Coh}(\mathcal{C})$ and choose presentations $\mathcal{Y}_c(X_i) \to \mathcal{Y}_c(Y_i) \to F_i \to 0$. Since φ^* is exact by (ii), we obtain a presentation $\varphi^*\mathcal{Y}_c(X_i) \to \varphi^*\mathcal{Y}_c(Y_i) \to \varphi^*F_i \to 0$ and hence an exact sequence

$$\bigoplus_i \varphi^*\mathcal{Y}_c(X_i) \to \bigoplus_i \varphi^*\mathcal{Y}_c(Y_i) \to \bigoplus_i \varphi^*(F_i) \to 0.$$

By Proposition 21.6 we obtain a presentation

$$\mathcal{Y}_c\Big(\bigoplus_i X_i\Big) \to \mathcal{Y}_c\Big(\bigoplus_i Y_i\Big) \to \bigoplus_i F_i \to 0.$$

By applying φ^* and using that $\varphi^*\mathcal{Y}_c$ commutes with countable direct sums, we obtain an exact sequence

$$\bigoplus_i \varphi^*\mathcal{Y}_c(X_i) \to \bigoplus_i \varphi^*\mathcal{Y}_c(Y_i) \to \varphi^*\Big(\bigoplus_i F_i\Big) \to 0.$$

We deduce that the map $\bigoplus_i \varphi^*(F_i) \xrightarrow{\sim} \varphi^*(\bigoplus_i F_i)$ is an isomorphism.

Step 2: $\varphi^*\colon \operatorname{Coh}(\mathcal{C}) \to \operatorname{Coh}(\mathcal{P})$ admits a fully faithful left adjoint $\varphi_!$ given by $\varphi_!(\operatorname{Hom}_{\mathcal{P}}(-,P)) = \operatorname{Hom}_{\mathcal{C}}(-,\varphi(P))$ for all $P \in \mathcal{P}$.

Observe that we have an adjunction $\varphi_!\colon \mathsf{Mod}(\mathcal{P}) \rightleftarrows \mathsf{Mod}(\mathcal{C}) :\varphi^*$, where $\varphi_!$ is given by left Kan extension along $\varphi^{\mathrm{op}}\colon \mathcal{P}^{\mathrm{op}} \to \mathcal{C}^{\mathrm{op}}$. As φ is fully faithful, so is $\varphi_!$ by Corollary 13.16. Moreover, the diagram

$$\begin{tikzcd} \mathcal{P} \arrow[r, "\varphi"] \arrow[d, "\mathcal{Y}_c"] & \mathcal{C} \arrow[d, "\mathcal{Y}_c"] \\ \mathsf{Mod}(\mathcal{P}) \arrow[r, dashed, "\exists!\varphi_!"'] & \mathsf{Mod}(\mathcal{C}) \end{tikzcd}$$

commutes, because for all $F \in \mathsf{Mod}(\mathcal{C})$ and $P \in \mathcal{P}$ we have natural isomorphisms

$$\operatorname{Hom}_{\mathsf{Mod}(\mathcal{C})}\big(\mathcal{Y}_c\varphi(P), F\big) \cong F(\varphi(P)) \cong \operatorname{Hom}\big(\mathcal{Y}_c(P), \varphi^*(F)\big) \cong \operatorname{Hom}\big(\varphi_!\mathcal{Y}_c(P), F\big),$$

and hence $\varphi_!\mathcal{Y}_c \cong \mathcal{Y}_c\varphi$ by the Yoneda lemma. It follows that $\varphi_!$ preserves coherent functors. Further, φ^* preserves coherent functors by (ii). Hence, we obtain an adjunction $\varphi_!\colon \operatorname{Coh}(\mathcal{P}) \rightleftarrows \operatorname{Coh}(\mathcal{C}) :\varphi^*$.

Step 3: End of the proof.

Observe that a coherent functor $F \in \operatorname{Coh}(\mathcal{C})$ lies in $\operatorname{Ker}(\varphi^*)$ if and only if F is the cokernel of $\mathcal{Y}_c(X) \to \mathcal{Y}_c(Y)$, for some map $X \to Y$, such that for all $P \in \mathcal{P}$ the induced map $\operatorname{Hom}_{\mathcal{C}}(P,X) \twoheadrightarrow \operatorname{Hom}_{\mathcal{C}}(P,Y)$ is surjective. We conclude with the following equivalences:

$$\begin{aligned} \text{(PG2) holds} &\iff \operatorname{Ker}(\varphi^*) \text{ is closed under countable direct sums} \\ &\iff \varphi^*\colon \operatorname{Coh}(\mathcal{C}) \to \operatorname{Coh}(\mathcal{P}) \text{ preserves countable direct sums} \qquad \text{(Exercise 9.7)} \\ &\iff \mathcal{C} \to \operatorname{Coh}(\mathcal{P}) \text{ preserves countable direct sums.} \end{aligned}$$

$\square$

Theorem 23.3 (Brown Representability)**.** *Let $(\mathcal{C}, T)$ be a triangulated category with arbitrary direct sums. Suppose that $\mathcal{C}$ admits a set $\mathcal{P}_0$ of perfect generators.*

(i) If $\mathcal{L} \subseteq \mathcal{C}$ is a triangulated subcategory which is closed under arbitrary direct sums and contains $\mathcal{P}_0$, then $\mathcal{L} = \mathcal{C}$.

(ii) Let $H\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$ be a cohomological functor which preserves arbitrary products (i.e., H sends arbitrary direct sums in $\mathcal{C}$ to products in Ab). Then H is representable, that is, there exists $X \in \mathcal{C}$ and a natural isomorphism

$$\operatorname{Hom}_{\mathcal{C}}(-,X) \xrightarrow{\sim} H$$

of functors $\mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$.

Proof. Replacing $\mathcal{P}_0$ by $\bigcup_{n\in\mathbb{Z}} T^n(\mathcal{P}_0)$, we may assume $T^{\pm 1}(\mathcal{P}_0) = \mathcal{P}_0$. We denote by $\mathcal{P} \subseteq \mathcal{C}$ the full subcategory spanned by all direct sums of the P for $P \in \mathcal{P}_0$. Let $\mathcal{L} \subseteq \mathcal{C}$ be the smallest triangulated subcategory which is closed under all direct sums and contains $\mathcal{P}_0$.

Let $H\colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$ be a cohomological functor which preserves arbitrary products. We will inductively construct maps $f_i\colon X_i \to X_{i+1}$ in $\mathcal{L}$ and $\phi_i\colon \mathcal{Y}(X_i) \to H$ in $\mathsf{Mod}(\mathcal{C})$ such that for all $i \in \mathbb{Z}_{\geq 0}$ we have:

(a) $\phi_i = \phi_{i+1} \circ \mathcal{Y}(f_i)$;

(b) The map $\phi_{i+1,P}\colon \operatorname{Hom}_{\mathcal{C}}(P, X_{i+1}) \twoheadrightarrow H(P)$ is surjective, for all $P \in \mathcal{P}_0$;

(c) $\operatorname{Ker}(\operatorname{Hom}_{\mathcal{C}}(P, X_i) \xrightarrow{f_{i*}} \operatorname{Hom}_{\mathcal{C}}(P, X_{i+1})) = \operatorname{Ker}(\phi_{iP})$, for all $P \in \mathcal{P}_0$.

We put $X_0 = 0$ and let $\phi_0\colon \mathcal{Y}(X_0) \to H$ be the zero map. We define $X_1 \coloneqq \bigoplus_{P\in\mathcal{P}_0} P^{\oplus H(P)} \in \mathcal{L}$. Observe that

$$\operatorname{Hom}(\mathcal{Y}(X_1), H) = H(X_1) = \prod_{P\in\mathcal{P}_0} \prod_{H(P)} H(P) = \prod_{P\in\mathcal{P}_0} \operatorname{Hom}_{\mathsf{Set}}(H(P), H(P)),$$

and so we let $\phi_1\colon \mathcal{Y}(X_1) \to H$ be the map corresponding to the identity on $H(P)$ for each $P \in \mathcal{P}_0$. Note that the map $\phi_{1P}\colon \operatorname{Hom}_{\mathcal{C}}(P, X_1) \twoheadrightarrow H(P)$ is surjective by construction, for each $P \in \mathcal{P}_0$.

Suppose that we have constructed X_i, and maps $f_{i-1}\colon X_{i-1} \to X_i$, $\phi_i\colon \mathcal{Y}(X_i) \to H$ satisfying (a), (b) and (c), for all $i < n$. Consider the object $K_n \coloneqq \bigoplus_{P\in\mathcal{P}_0} P^{\oplus \operatorname{Ker}(\phi_n)(P)} \in \mathcal{L}$ and the map $k_n\colon K_n \to X_n$ given by $f\colon P \to X_n$ on the summand corresponding to $f \in \operatorname{Ker}(\phi_n)(P) \subseteq \operatorname{Hom}(P, X_n)$. Observe that the composite $\mathcal{Y}(K_n) \to \mathcal{Y}(X_n) \xrightarrow{\phi_n} H$ vanishes, because the corresponding element in $H(K_n) = \prod_{P\in\mathcal{P}_0} \prod_{\operatorname{Ker}(\phi_n)(P)} H(P)$ is zero. In other words, k_n factors as

$$\mathcal{Y}(K_n) \to \operatorname{Ker}(\phi_n) \to \mathcal{Y}(X_n).$$

We now complete $k_n\colon K_n \to X_n$ to a distinguished triangle

$$K_n \xrightarrow{k_n} X_n \xrightarrow{f_n} X_{n+1} \to T(K_n) \tag{23.1}$$

in $\mathcal{L}$. Since H is cohomological, we obtain a commutative diagram

$$\begin{tikzcd}
H(X_{n+1}) \arrow[r, "H(f_n)"] \arrow[d, equal] & H(X_n) \arrow[rr, "H(k_n)"] \arrow[d, equal] & & H(K_n) \arrow[d, equal] \\
\operatorname{Hom}(\mathcal{Y}(X_{n+1}), H) \arrow[r] & \operatorname{Hom}(\mathcal{Y}(X_n), H) \arrow[r] & \operatorname{Hom}(\operatorname{Ker}(\phi_n), H) \arrow[r] & \operatorname{Hom}(\mathcal{Y}(K_n), H)
\end{tikzcd}$$

in Ab, where the top row is exact. Identifying $\phi_n\colon \mathcal{Y}(X_n) \to H$ with an element of $H(X_n)$, we deduce $H(k_n)(\phi_n) = 0$ from the fact that the composite $\operatorname{Ker}(\phi_n) \to \mathcal{Y}(X_n) \xrightarrow{\phi_n} H$ vanishes. Hence, we find $\phi_{n+1} \in \operatorname{Hom}(\mathcal{Y}(X_{n+1}), H) = H(X_{n+1})$ such that $\phi_{n+1} \circ \mathcal{Y}(f_n) = H(f_n)(\phi_{n+1}) = \phi_n$. Thus, we have constructed maps $f_n\colon X_n \to X_{n+1}$ and $\phi_{n+1}\colon \mathcal{Y}(X_{n+1}) \to H$ such that (a) is satisfied. Then (b) and the inclusion "$\subseteq$" in (c) are a formal consequence of (a). For the converse inclusion in (c), we apply the cohomological functor $\operatorname{Hom}_{\mathcal{C}}(P, -)$ (where $P \in \mathcal{P}_0$) to the distinguished triangle (23.1) to obtain an exact sequence

$$\operatorname{Hom}_{\mathcal{C}}(P, K_n) \xrightarrow{k_{n*}} \operatorname{Hom}_{\mathcal{C}}(P, X_n) \xrightarrow{f_{n*}} \operatorname{Hom}_{\mathcal{C}}(P, X_{n+1}).$$

It therefore suffices to show that $\mathrm{Ker}(\phi_{nP}) \subseteq \mathrm{Im}(k_{n*})$, for any $P \in \mathcal{P}_0$. To this end, let $\alpha \in \mathrm{Ker}(\phi_{nP}) = \mathrm{Ker}(\phi_n)(P) \subseteq \mathrm{Hom}_{\mathcal{C}}(P, X_n)$. Denoting by $\widetilde{\alpha}\colon P \to \bigoplus_{P' \in \mathcal{P}_0} P'^{\oplus \mathrm{Ker}(\phi_n)(P')} = K_n$ the inclusion into the component corresponding to α, we obtain $k_n \circ \widetilde{\alpha} = \alpha$ as desired. This completes the construction.

Consider the distinguished triangle

$$\bigoplus_{i\geq 0} X_i \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i\geq 0} X_i \to X \to T(\bigoplus_{i\geq 0} X_i), \tag{23.2}$$

so that $X = \mathrm{hocolim}_i X_i \in \mathcal{L}$. Since H is cohomological and commutes with countable products, we obtain an exact sequence

$$H(X) \to \prod_{i=0}^{\infty} H(X_i) \xrightarrow{\mathrm{id}-\mathrm{shift}} \prod_{i=0}^{\infty} H(X_i).$$

Viewing $\phi_i\colon \mathcal{Y}(X_i) \to H$ as an element of $H(X_i)$, we observe that $(\phi_i)_i$ is killed by $\mathrm{id}-\mathrm{shift}$. Hence, there exists $\phi \in H(X)$ mapping to $(\phi_i)_i$. In other words, the $\phi_i\colon \mathcal{Y}(X_i) \to H$ give rise to a map

$$\phi\colon \mathcal{Y}(X) \to H.$$

Note that by (a)–(c) there are commutative diagrams

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(\phi_i)\big|_{\mathcal{P}} \arrow[r] \arrow[d, "0"'] & \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}} \arrow[r, "\phi_i"] \arrow[d, "f_{i*}"] & H\big|_{\mathcal{P}} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{Ker}(\phi_{i+1})\big|_{\mathcal{P}} \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(-, X_{i+1})\big|_{\mathcal{P}} \arrow[r, "\phi_{i+1}"'] & H\big|_{\mathcal{P}} \arrow[r] & 0.
\end{tikzcd}$$

By passing to direct sums, and noting that $\bigoplus_i \mathrm{Ker}(\phi_i)\big|_{\mathcal{P}} \hookrightarrow \bigoplus_i \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}}$ is a monomorphism as $\mathrm{Coh}(\mathcal{C})$ is AB4 by Corollary 22.4 and $\mathrm{Coh}(\mathcal{C}) \to \mathrm{Coh}(\mathcal{P})$ is exact and preserves countable direct sums by Lemma 23.2, we obtain a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathrm{Ker}(\mathrm{id}-\mathrm{shift}) \arrow[r] \arrow[d, hook] & 0 \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=0}^{\infty} \mathrm{Ker}(\phi_i)\big|_{\mathcal{P}} \arrow[r] \arrow[d, "\mathrm{id}-0"', "\simeq"] & \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}} \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{shift}"] & \bigoplus_{i=0}^{\infty} H\big|_{\mathcal{P}} \arrow[r] \arrow[d, "\mathrm{id}-\mathrm{shift}"] & 0 \\
0 \arrow[r] & \bigoplus_{i=0}^{\infty} \mathrm{Ker}(\phi_i)\big|_{\mathcal{P}} \arrow[r] \arrow[d] & \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}} \arrow[r] \arrow[d, two heads] & \bigoplus_{i=0}^{\infty} H\big|_{\mathcal{P}} \arrow[r] \arrow[d, two heads] & 0 \\
 & 0 \arrow[r] & \mathrm{Coker}(\mathrm{id}-\mathrm{shift}) \arrow[r] & H\big|_{\mathcal{P}} \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns. Hence, the middle column constitutes an exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}} \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-, X_i)\big|_{\mathcal{P}} \longrightarrow H\big|_{\mathcal{P}} \longrightarrow 0.$$

By Lemma 23.2 we obtain an isomorphism $\bigoplus_i \mathrm{Hom}_{\mathcal{C}}(-,X_i)\big|_{\mathcal{P}} \cong \mathrm{Hom}_{\mathcal{C}}(-,\bigoplus_i X_i)\big|_{\mathcal{P}}$. Therefore, applying the cohomological functor $\mathcal{C} \to \mathrm{Coh}(\mathcal{P})$ to (23.2) yields a short exact sequence

$$0 \longrightarrow \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-,X_i)\big|_{\mathcal{P}} \xrightarrow{\mathrm{id}-\mathrm{shift}} \bigoplus_{i=0}^{\infty} \mathrm{Hom}_{\mathcal{C}}(-,X_i)\big|_{\mathcal{P}} \longrightarrow \mathrm{Hom}_{\mathcal{C}}(-,X)\big|_{\mathcal{P}} \longrightarrow 0,$$

where we have used that $\mathcal{P}$ is closed under translation. Combining both short exact sequences, we deduce that

$$\phi\big|_{\mathcal{P}} \colon \mathrm{Hom}_{\mathcal{C}}(-,X)\big|_{\mathcal{P}} \xrightarrow{\sim} H\big|_{\mathcal{P}} \tag{23.3}$$

is an isomorphism. It remains to show that ϕ is an isomorphism. Let $\mathcal{T} \subseteq \mathcal{C}$ be the full subcategory spanned by the objects Y such that $\phi_{T^i(Y)} \colon \mathrm{Hom}_{\mathcal{C}}(T^i(Y),X) \to H(T^i(Y))$ is an isomorphism for all $i \in \mathbb{Z}$. Since both $\mathrm{Hom}_{\mathcal{C}}(-,X)$ and H are cohomological and commute with arbitrary products, it follows that $\mathcal{T}$ is a triangulated subcategory which is closed under arbitrary direct sums. Since also $\mathcal{P}_0 \subseteq \mathcal{T}$ by (23.3), we deduce $\mathcal{L} \subseteq \mathcal{T}$.

We are thus reduced to showing that $\mathcal{L} = \mathcal{C}$. To this end, let $Y \in \mathcal{C}$ be arbitrary. Applying the previous discussion to $H = \mathrm{Hom}_{\mathcal{C}}(-,Y)$, we construct a sequence $X_0 \xrightarrow{f_0} X_1 \xrightarrow{f_1} \cdots$ in $\mathcal{L}$ and a map $\widetilde{\phi} \colon X = \mathrm{hocolim}_i X_i \to Y$ with $X \in \mathcal{L}$ such that the map

$$\mathrm{Hom}_{\mathcal{C}}(T^i(P),X) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{C}}(T^i(P),Y)$$

is an isomorphism for all i and $P \in \mathcal{P}_0$. Completing $\widetilde{\phi}$ to a distinguished triangle $X \xrightarrow{\widetilde{\phi}} Y \to Z \to T(X)$, we deduce from the fact that $\mathrm{Hom}_{\mathcal{C}}(P,-)$ is cohomological that $\mathrm{Hom}_{\mathcal{C}}(P,Z) = 0$ for all i and $P \in \mathcal{P}_0$. From (PG1) we get $Z = 0$, hence $\widetilde{\phi} \colon X \to Y$ is an isomorphism. This shows $Y \in \mathcal{L}$ and hence $\mathcal{L} = \mathcal{C}$, which finishes the proof. ☐

We first look at some consequences of Brown representability before we turn to examples.

Corollary 23.4. *Let $(\mathcal{C},T)$ be a triangulated category with arbitrary direct sums, which admits a set $\mathcal{P}_0$ of perfect generators.*

(i) $\mathcal{C}$ admits arbitrary products.

(ii) Let $L \colon \mathcal{C} \to \mathcal{D}$ be an exact functor which commutes with arbitrary direct sums. Then L admits an exact right adjoint.

Proof. For part (i), let $\{X_i\}_i$ be a family of objects in $\mathcal{C}$. We apply Theorem 23.3 to the cohomological functor

$$H \coloneqq \prod_i \mathrm{Hom}_{\mathcal{C}}(-,X_i) \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}.$$

We deduce that there exists $X \in \mathcal{C}$ such that $\mathrm{Hom}_{\mathcal{C}}(Y,X) = \prod_i \mathrm{Hom}_{\mathcal{C}}(Y,X_i)$ naturally in $Y \in \mathcal{C}$. In other words, $X = \prod_i X_i$.

For part (ii), we note that the functor $\mathrm{Hom}_{\mathcal{D}}(L(-),D) \colon \mathcal{C}^{\mathrm{op}} \to \mathsf{Ab}$ is cohomological and commutes with arbitrary direct sums. Hence, by Theorem 23.3 there exists $R(D) \in \mathcal{C}$ and a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}}\big(-,R(D)\big) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{D}}(L(-),D).$$

Then the $R(D)$ assemble into a right adjoint $R\colon \mathcal{D} \to \mathcal{C}$ of L. Finally, R is exact by Proposition 6.5. □

Definition 23.5. Let $\mathcal{C}$ be a triangulated category. An object $X \in \mathcal{C}$ is called *compact* if the functor $\mathrm{Hom}_{\mathcal{C}}(X, \_)\colon \mathcal{C} \to \mathsf{Ab}$ commutes with arbitrary direct sums.

Corollary 23.6. *Let $(\mathcal{C}, T)$ be a triangulated category which admits arbitrary direct sums. Suppose that $\mathcal{C}$ admits a set $\mathcal{P}_0 \subseteq \mathcal{C}$ of* compact generators, *i.e., a set of generators which are compact.*

Then both $\mathcal{C}$ and $\mathcal{C}^{\mathrm{op}}$ satisfy the hypotheses of Theorem 23.3.

Proof. We call a set $\mathcal{S}_0 \subseteq \mathcal{C}$ a set of *symmetric generators* if $\mathcal{S}_0$ satisfies (PG1) and

(PG3) there exists a set $\mathcal{T}_0 \subseteq \mathcal{C}$ such that for every map $X \to Y$ in $\mathcal{C}$ the map $\mathrm{Hom}_{\mathcal{C}}(S, X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}(S, Y)$ is surjective for all $S \in \mathcal{S}_0$ if and only if $\mathrm{Hom}_{\mathcal{C}}(Y, T) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, T)$ is injective for all $T \in \mathcal{T}_0$.

Note that (PG3) implies (PG2). We deduce that if $\mathcal{S}_0$ is a set of symmetric generators, then $\mathcal{S}_0$ (resp. $\mathcal{T}_0$) is a set of perfect generators in $\mathcal{C}$ (resp. $\mathcal{C}^{\mathrm{op}}$).[1] It therefore suffices to prove that every set of compact generators satisfies (PG3).

By the compactness it is clear that $\mathcal{P}_0$ is a set of perfect generators, and hence Brown representability holds for $\mathcal{C}$. Hence, $\mathcal{C}^{\mathrm{op}}$ admits arbitrary direct sums by Corollary 23.4. Fix any $P \in \mathcal{P}_0$. Then the functor

$$\begin{aligned} H\colon \mathcal{C}^{\mathrm{op}} &\to \mathsf{Ab}, \\ X &\mapsto \mathrm{Hom}_{\mathsf{Ab}}\big(\mathrm{Hom}_{\mathcal{C}}(P, X), \mathbb{Q}/\mathbb{Z}\big) \end{aligned}$$

is cohomological (since $\mathbb{Q}/\mathbb{Z}$ is injective in Ab) and preserves arbitrary products (since P is compact). By Theorem 23.3 we find $Q_P \in \mathcal{C}$ and a natural isomorphism $\mathrm{Hom}_{\mathcal{C}}(\_, Q_P) \xrightarrow{\sim} H$. Since $\mathbb{Q}/\mathbb{Z}$ is a cogenerator for Ab, we deduce that for any map $X \to Y$, the map $\mathrm{Hom}_{\mathcal{C}}(P, X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}}(P, Y)$ is surjective if and only if $\mathrm{Hom}_{\mathcal{C}}(Y, Q_P) \hookrightarrow \mathrm{Hom}_{\mathcal{C}}(X, Q_P)$ is injective. Since P was arbitrary, it follows that $\mathcal{P}_0$ satisfies (PG3). □

Example 23.7. Let Λ be a ring. Then $\mathsf{Mod}(\Lambda)$ is AB4* and $\mathsf{D}(\mathsf{Mod}(\Lambda))$ is compactly generated by Λ: Indeed, Λ generates because $\mathrm{Hom}_{\mathsf{D}(\mathsf{Mod}(\Lambda))}(\Lambda, X) = \mathrm{H}^0(X)$ for every $X \in \mathsf{D}(\mathsf{Mod}(\Lambda))$; and Λ is compact, because the formation of arbitrary direct sums in $\mathsf{Mod}(\Lambda)$ is exact, and hence H^0 commutes with arbitrary direct sums.

From Brown representability we obtain an alternative proof of Theorem 17.11:

Theorem 23.8. *Let $\mathcal{A}$ be an abelian category which satisfies AB4*. Suppose that $\mathcal{A}$ admits a set $\mathcal{I}$ of injective cogenerators.*[2] *We write $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \subseteq \mathsf{K}(\mathcal{A})$ for the smallest triangulated subcategory containing $\mathcal{I}$ and which is closed under arbitrary products.*

Then the inclusion $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \hookrightarrow \mathsf{K}(\mathcal{A})$ admits a left adjoint $\mathbf{i}\colon \mathsf{K}(\mathcal{A}) \to \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ and the composite $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \to \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an equivalence of categories.

---

[1] To see that $\mathcal{T}_0$ is a symmetric set of generators of $\mathcal{C}^{\mathrm{op}}$, use that, if $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} X[1]$ is a distinguished triangle in $\mathcal{C}^{\mathrm{op}}$, then $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(T, X) \twoheadrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(T, Y)$ is surjective if and only if $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(T, Z) \hookrightarrow \mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(T, X[1])$ is injective.

[2] This means that every object of $\mathcal{A}$ embeds into a product of objects in $\mathcal{I}$. In particular $\mathrm{Hom}_{\mathcal{A}}(X, I) = 0$ for all $I \in \mathcal{I}$ implies $X = 0$.

Proof. We first show that for any $I \in \mathcal{I}$ we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X, I) \cong \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(X), I) \tag{23.4}$$

in $X \in \mathsf{K}(\mathcal{A})$. We consider the commutative diagram

$$\begin{tikzcd}
B^0(X,I) \arrow[r] \arrow[d, "\simeq"'] & \mathrm{Hom}_{\mathsf{C}(\mathcal{A})}(X,I) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X,I) \arrow[r] \arrow[d] & 0 \\
\{f \mid f(\mathrm{H}^0(X)) = 0\} \arrow[r, hook] & \mathrm{Hom}_{\mathcal{A}}\big(X^0/\operatorname{Im}(d^{-1}), I\big) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}(\mathrm{H}^0(X), I) \arrow[r] & 0,
\end{tikzcd}$$

where $B^0(X, I)$ denotes the subgroup of null homotopic maps, and the lower row is exact since I is injective. Then the right vertical map is an isomorphism by the five lemma.

We claim that $\mathcal{I} \subseteq \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ is a set of perfect cogenerators, *i.e.*, a set of perfect generators in $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})^{\mathrm{op}}$. Indeed, let $X \in \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ such that $\mathrm{Hom}(X, I[n]) = 0$ for all $n \in \mathbb{Z}$ and $I \in \mathcal{I}$. Then the full subcategory $\mathcal{T} \subseteq \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ spanned by the objects Y such that $\mathrm{Hom}(X, Y[n]) = 0$ for all $n \in \mathbb{Z}$ is triangulated and closed under arbitrary products. As $\mathcal{T}$ contains $\mathcal{I}$, we deduce $\mathcal{T} = \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. Now the Yoneda lemma implies $X = 0$. Hence, $\mathcal{I}$ is a set of cogenerators for $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. Let now $\{X_i \to Y_i\}_i$ be a countable family of maps in $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ such that $\mathrm{Hom}(Y_i, I) \twoheadrightarrow \mathrm{Hom}(X_i, I)$ is surjective for all $I \in \mathcal{I}$ and all i. Since $\mathcal{I}$ is a set of cogenerators, the isomorphism (23.4) shows that $\mathrm{H}^0(X_i) \hookrightarrow \mathrm{H}^0(Y_i)$ is a monomorphism. As H^0 commutes with arbitrary products and the formation of products is left exact, we deduce that $\mathrm{H}^0(\prod_i X_i) \hookrightarrow \mathrm{H}^0(\prod_i Y_i)$ is a monomorphism. Hence, in the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})}\Big(\prod_i Y_i, I\Big) \arrow[r] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})}\Big(\prod_i X_i, I\Big) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathcal{A}}\Big(\mathrm{H}^0\Big(\prod_i Y_i\Big), I\Big) \arrow[r, two heads] & \mathrm{Hom}_{\mathcal{A}}\Big(\mathrm{H}^0\Big(\prod_i X_i\Big), I\Big)
\end{tikzcd}$$

the bottom horizontal map is surjective. It follows that the top horizontal map is is surjective, *i.e.*, $\mathcal{I}$ satisfies (PG2). Hence, $\mathcal{I}$ is a set of perfect cogenerators in $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$.

Since the inclusion $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \hookrightarrow \mathsf{K}(\mathcal{A})$ preserves arbitrary products, it admits a left adjoint $\mathbf{i}$ by Corollary 23.4. We denote by $\eta\colon \mathrm{id}_{\mathsf{K}(\mathcal{A})} \to \mathbf{i}$ the unit of the adjunction. Let $X \in \mathsf{K}(\mathcal{A})$. We claim that $\eta_X\colon X \to \mathbf{i}X$ is a quasi-isomorphism. For each $I \in \mathcal{I}$ and $n \in \mathbb{Z}$ we have a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathsf{K}(\mathcal{A})}\big(\mathbf{i}X[n], I\big) \arrow[r, "\sim"] \arrow[d, "\sim"'] & \mathrm{Hom}_{\mathsf{K}(\mathcal{A})}(X[n], I) \arrow[d, "\sim"] \\
\mathrm{Hom}_{\mathcal{A}}\big(\mathrm{H}^n(\mathbf{i}X), I\big) \arrow[r] & \mathrm{Hom}_{\mathcal{A}}\big(\mathrm{H}^n(X), I\big),
\end{tikzcd}$$

where the top horizontal map comes from the adjunction and the vertical maps are (23.4). We deduce that the bottom horizontal map is an isomorphism. As $\mathcal{I}$ is a set of cogenerators, it follows that $\mathrm{H}^n(X) \xrightarrow{\sim} \mathrm{H}^n(\mathbf{i}X)$ is an isomorphism for all n. Hence, η_X is a quasi-isomorphism.

Let now $f\colon X \to Y$ be a quasi-isomorphism in $\mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$. We claim that f is an isomorphism. Let $\mathcal{T} \subseteq \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$ be the full subcategory spanned by the objects Z such that $f^*\colon \mathrm{Hom}(Y, Z[n]) \xrightarrow{\sim}$

$\mathrm{Hom}(X, Z[n])$ is an isomorphism for all $n \in \mathbb{Z}$. Then $\mathcal{T}$ is triangulated and closed under products. From (23.4) we deduce $\mathcal{I} \subseteq \mathcal{T}$, because f is a quasi-isomorphism. It follows that $\mathcal{T} = \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})$, and then the Yoneda lemma implies that f is an isomorphism.

We now apply Corollary 10.9 (where we observe that $(*)$ is satisfied since η is a quasi-isomorphism) to deduce that the functor

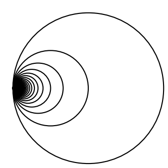
$$\mathsf{K}_{\mathrm{hinj}}(\mathcal{A}) \cong \mathsf{K}_{\mathrm{hinj}}(\mathcal{A})_{\mathrm{qis}} \xrightarrow{\sim} \mathsf{K}(\mathcal{A})_{\mathrm{qis}} = \mathsf{D}(\mathcal{A})$$

is an equivalence of categories. □

§24. Grothendieck categories

Let $F\colon \mathcal{A} \to \mathcal{B}$ be a (left exact) functor between abelian categories. In Theorem 17.11 we found that F extends to a derived functor on the unbounded derived categories if $\mathcal{A}$ is AB4*, *i.e.*, arbitrary products exist and are exact. However, many abelian categories of interest are not AB4* (*e.g.*, sheaves on a topological space, quasi-coherent sheaves on a scheme, ...). Hence it is desirable to prove the existence of unbounded derived functors from weaker conditions on $\mathcal{A}$.

Example 24.1. Let X be the Hawaiian earring:



Then $\mathrm{Shv}(X, \mathsf{Ab})$ is an abelian category which does not satisfy AB4*. See [Wu] for details.

As it turns out, basically all abelian categories of interest are Grothendieck categories.

Definition 24.2. Let $\mathcal{A}$ be an abelian category.

(i) An object $G \in \mathcal{A}$ is called a *generator* if the functor $\mathrm{Hom}_{\mathcal{A}}(G, -)\colon \mathcal{A} \to \mathsf{Ab}$ is conservative.

(ii) An abelian category $\mathcal{A}$ is called *Grothendieck* if it is AB5 (filtered colimits exist and are exact) and has a generator.

Lemma 24.3. *Let $\mathcal{A}$ be an abelian AB3 category (i.e., $\mathcal{A}$ admits arbitrary direct sums). For an object $G \in \mathcal{A}$ the following conditions are equivalent:*

(a) G is a generator.

(b) The functor $\mathrm{Hom}_{\mathcal{A}}(G, -)\colon \mathcal{A} \to \mathsf{Ab}$ is faithful.

(c) The functor $\mathrm{Hom}_{\mathcal{A}}(G, -)$ detects epimorphisms and monomorphisms.

(d) For every object $A \in \mathcal{A}$ the canonical map $G^{\oplus \mathrm{Hom}(G,A)} \twoheadrightarrow A$ is an epimorphism.

Proof. Suppose that G is a generator. Let $f\colon A \to B$ be a map such that $\mathrm{Hom}_{\mathcal{A}}(G, f) = 0$. Denote by $\iota\colon \mathrm{Ker}(f) \to A$ the inclusion of the kernel of f. Since $\mathrm{Hom}_{\mathcal{A}}(G, -)$ is left exact, we obtain an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{A}}(G, \mathrm{Ker}(f)) \xrightarrow{\iota_*} \mathrm{Hom}_{\mathcal{A}}(G, A) \xrightarrow{0} \mathrm{Hom}_{\mathcal{A}}(G, B).$$

It follows that ι_* is an isomorphism. Since G is a generator, ι is an isomorphism, which shows $f = 0$. Hence $\mathrm{Hom}_{\mathcal{A}}(G, -)$ is faithful.

For the implication "(b) $\Longrightarrow$ (c)" let $f\colon A \to B$ be a map such that $f_*\colon \mathrm{Hom}_{\mathcal{A}}(G, A) \to \mathrm{Hom}_{\mathcal{A}}(G, B)$ is an epimorphism. Let $p, q\colon B \rightrightarrows C$ be two maps with $pf = qf$. Then $p_* f_* = q_* f_*$ and hence $p_* = q_*$ as maps $\mathrm{Hom}_{\mathcal{A}}(G, B) \to \mathrm{Hom}_{\mathcal{A}}(G, C)$ since f_* is an epimorphism. But then $p = q$ since $\mathrm{Hom}_{\mathcal{A}}(G, -)$ is faithful, which shows that f is an epimorphism. A similar argument shows that, if f_* is a monomorphism, then so is f.

The implication "(c) $\Longrightarrow$ (d)" is clear.

We show "(d) $\Longrightarrow$ (b)". Let $f, g\colon A \rightrightarrows B$ such that $f_*, g_*\colon \mathrm{Hom}_{\mathcal{A}}(G, A) \rightrightarrows \mathrm{Hom}_{\mathcal{A}}(G, B)$ agree. In other words, we have $f\varphi = g\varphi$ for every $\varphi\colon G \to A$. But then also $f\pi = g\pi$, where π denotes the epimorphism $G^{\oplus \mathrm{Hom}(G,A)} \to A$. We deduce that $f = g$ and hence $\mathrm{Hom}_{\mathcal{A}}(G, -)$ is faithful.

Finally, the implication "(c) $\Longrightarrow$ (a)" follows from the fact that a morphism in an abelian category is an isomorphism as soon as it is an epimorphism and a monomorphism. □

Remark 24.4. We analyze the conditions under which the implications in Lemma 24.3 hold. We fix a category $\mathcal{C}$.

The implication "(a) $\Longrightarrow$ (b)" holds if $\mathcal{C}$ admits equalizers, "(b) $\Longrightarrow$ (c)" always holds, "(c) $\Longleftrightarrow$ (d)" holds if $\mathcal{C}$ admits arbitrary coproducts, and "(c) $\Longrightarrow$ (a)" holds if a map in $\mathcal{C}$ is an isomorphism as soon as it is monic and epic.

Lemma 24.5. *Let A be an object in a Grothendieck category $\mathcal{A}$. Then the subobjects of A form a set.*

Proof. Let $G \in \mathcal{A}$ be a generator. Then $\mathrm{Hom}_{\mathcal{A}}(G, A)$ is a set, and it suffices to show that the class of subobjects of A embedds into the power set of $\mathrm{Hom}_{\mathcal{A}}(G, A)$. Let $f_i\colon U_i \hookrightarrow A$ be subobjects $(i = 1, 2)$. Put $U \coloneqq U_1 \times_A U_2$ and consider the induced cartesian diagram

$$\begin{tikzcd}
\mathrm{Hom}_{\mathcal{A}}(G, U) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Hom}_{\mathcal{A}}(G, U_2) \arrow[d, hook, "f_{2*}"] \\
\mathrm{Hom}_{\mathcal{A}}(G, U_1) \arrow[r, hook, "f_{1*}"'] & \mathrm{Hom}_{\mathcal{A}}(G, A).
\end{tikzcd}$$

If the images of f_{1*} and f_{2*} coincide, then the top horizontal and the left vertical maps are isomorphisms. Since G is a generator, we conclude that the projections $U \xrightarrow{\sim} U_i$ are isomorphisms, hence U_1 and U_2 are equivalent subobjects of A. □

Theorem 24.6 (Gabriel–Popescu)**.** *Let $\mathcal{A}$ be a Grothendieck abelian category. Then there exists a ring Λ and an exact Bousfield localization*

$$T\colon \mathsf{Mod}(\Lambda) \to \mathcal{A}.$$

Proof. Let $G \in \mathcal{A}$ be a generator and put $\Lambda \coloneqq \mathrm{End}_{\mathcal{A}}(G)^{\mathrm{op}}$. Consider the functor

$$\begin{aligned} H\colon \mathcal{A} &\to \mathsf{Mod}(\Lambda), \\ A &\mapsto H(A) \coloneqq \mathrm{Hom}_{\mathcal{A}}(G, A), \end{aligned}$$

where Λ acts on $H(A)$ by functoriality through its action on G.

Step 1: H admits a left adjoint.

Let $\mathcal{C} \subseteq \mathsf{Mod}(\Lambda)$ be the full subcategory spanned by the Λ-modules M such that the functor

$$\begin{aligned} F_M \colon \mathcal{A} &\to \mathsf{Set}, \\ A &\mapsto \operatorname{Hom}_\Lambda(M, H(A)) \end{aligned}$$

is corepresentable, that is, there exists $T(M) \in \mathcal{A}$ and a natural isomorphism

$$\tau_M \colon \operatorname{Hom}_{\mathcal{A}}(T(M), \_) \xrightarrow{\sim} F_M.$$

We have $\Lambda \in \mathcal{C}$, Since we have a natural isomorphism

$$\operatorname{Hom}_{\mathcal{A}}(G, A) = H(A) \cong \operatorname{Hom}_\Lambda(\Lambda, H(A)) = F_\Lambda(A).$$

We show that $\mathcal{C}$ is closed under arbitrary direct sums. Let $\{M_i\}_{i\in I}$ be a family of Λ-modules in $\mathcal{C}$. Let $\tau_{M_i} \colon \operatorname{Hom}_{\mathcal{A}}(T(M_i), \_) \xrightarrow{\sim} F_{M_i}$ be natural isomorphisms. Then we have a natural isomorphism

$$\operatorname{Hom}_{\mathcal{A}}\Big(\bigoplus_{i\in I} T(M_i), \_\Big) = \prod_{i\in I} \operatorname{Hom}_{\mathcal{A}}(T(M_i), \_) \xrightarrow[\sim]{\prod \tau_{M_i}} \prod_{i \in I} \operatorname{Hom}_\Lambda(M_i, H(\_)) = \operatorname{Hom}_\Lambda\Big(\bigoplus_{i\in I} M_i, H(\_)\Big),$$

which shows $\bigoplus_i M_i \in \mathcal{C}$.

Next, $\mathcal{C}$ is also closed under cokernels. Let $f \colon M \to N$ be a map in $\mathcal{C}$ and denote by $\tau_M \colon \operatorname{Hom}_{\mathcal{A}}(T(M), \_) \xrightarrow{\sim} F_M$ and $\operatorname{Hom}_{\mathcal{A}}(T(N), \_) \xrightarrow{\sim} F_N$ the natural isomorphisms. For every $A \in \mathcal{A}$ we have a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_{\mathcal{A}}\big(\operatorname{Coker}(T(f)), A\big) \arrow[r] \arrow[d] & \operatorname{Hom}_{\mathcal{A}}(T(N), A) \arrow[r] \arrow[d, "\sim"', "\tau_N"] & \operatorname{Hom}_{\mathcal{A}}(T(M), A) \arrow[d, "\sim"', "\tau_M"] \\ 0 \arrow[r] & \operatorname{Hom}_\Lambda(\operatorname{Coker}(f), H(A)) \arrow[r] & \operatorname{Hom}_\Lambda(N, H(A)) \arrow[r] & \operatorname{Hom}_\Lambda(M, H(A)). \end{tikzcd}$$

By the five lemma, the left vertical map is an isomorphism, naturally in A. Therefore, $\operatorname{Coker}(f) \in \mathcal{C}$ as desired.

Since every Λ-module M is the cokernel of a map $\Lambda^{\oplus J} \to \Lambda^{\oplus I}$, and $\mathcal{C}$ contains Λ and is closed under arbitrary direct sums and cokernels, we deduce $M \in \mathcal{C}$. Therefore, $\mathcal{C} = \mathsf{Mod}(\Lambda)$. By construction, the assignments $M \mapsto T(M)$ assemble into a left adjoint

$$\begin{aligned} T \colon \mathsf{Mod}(\Lambda) &\to \mathcal{A}, \\ M &\mapsto G \underset{\Lambda}{\otimes} M. \end{aligned}$$

of H.

Step 2: If a map $f \colon M \to H(A)$ of Λ-modules is a monomorphism, then the adjoint $\phi \coloneqq \varepsilon_A \circ T(f) \colon T(M) \to TH(A) \to A$ is a monomorphism as well.

Observe first that the counit $\varepsilon_{G^{\oplus n}} \colon TH(G^{\oplus n}) \xrightarrow{\sim} G^{\oplus n}$ is an isomorphism for all $n \geq 0$. We have a commutative diagram

$$\begin{tikzcd} \operatorname{Hom}_\Lambda(\Lambda, \Lambda^{\oplus n}) \arrow[r, equal] & \operatorname{Hom}_\Lambda\big(\Lambda, H(G^{\oplus n})\big) \arrow[r, "T"] \arrow[dr, "\sim"'] & \operatorname{Hom}_{\mathcal{A}}\big(T(\Lambda), TH(G^{\oplus n})\big) \arrow[d, "\varepsilon_{G^{\oplus n}}"] \\ & & \operatorname{Hom}_{\mathcal{A}}\big(G, G^{\oplus n}\big). \end{tikzcd}$$

We deduce that every map $G \to G^{\oplus n}$ is of the form $T(v)$ for some $v\colon \Lambda \to \Lambda^{\oplus n}$.

We prove the claim by contraposition. So let $f\colon M \to H(A)$ be a map such that $K = \operatorname{Ker}(\phi) \subseteq T(M)$ is non-zero. We will show that f is not a monomorphism. Choose an epimorphism $\pi\colon \Lambda^{\oplus I} \twoheadrightarrow M$. Since filtered colimits are exact in $\mathcal{A}$, the natural map

$$\varinjlim_{\substack{J\subseteq I\\ \text{finite}}} \left(G^{\oplus J} \times_{T(M)} K\right) \xrightarrow{\sim} G^{\oplus I} \times_{T(M)} K$$

is an isomorphism. We therefore find a finite subset $J \subseteq I$ and a map $u\colon G \to G^{\oplus J}$ such that the composite

$$G \xrightarrow{u} G^{\oplus J} \xrightarrow{T(\pi)} T(M)$$

is non-zero and factors through K. By the observation above, we have $u = T(v)$ for some $v\colon \Lambda \to \Lambda^{\oplus J}$. Now we have a commutative diagram

$$\begin{tikzcd}
HT(\Lambda) \arrow[r, "HT(v)"] & HT(\Lambda^{\oplus J}) \arrow[r, "HT(\pi)"] & HT(M) \arrow[r, "H(\phi)"] & H(A) \\
\Lambda \arrow[u, "\eta_\Lambda"] \arrow[r, "v"'] & \Lambda^{\oplus J} \arrow[u, "\eta_{\Lambda^{\oplus J}}"] \arrow[r, "\pi"'] & M, \arrow[u, "\eta_M"] \arrow[ur, "f"'] &
\end{tikzcd}$$

from which we deduce $v\pi \neq 0$ and $fv\pi = 0$. Hence, f is not a monomorphism.

Step 3: The functor $H\colon \mathcal{A} \hookrightarrow \mathsf{Mod}(\Lambda)$ is fully faithful.

We need to show that the counit $\varepsilon_A\colon TH(A) \xrightarrow{\sim} \mathrm{id}_A$ is an isomorphism for every $A \in \mathcal{A}$. Note that the triangle identity gives $H\varepsilon_A \circ \eta_{H(A)} = \mathrm{id}_{H(A)}$. Hence $H\varepsilon_A = \operatorname{Hom}_{\mathcal{A}}(G, \varepsilon_A)$ is an epimorphism, and then Lemma 24.3 shows that ε_A is an epimorphism. Moreover, the identity $H(A) \to H(A)$ is a monomorphism, and hence Step 2 shows that ε_A is a monomorphism as well. Since $\mathcal{A}$ is abelian, it follows that ε_A is a natural isomorphism as desired.

Step 4: $T\colon \mathsf{Mod}(\Lambda) \to \mathcal{A}$ is exact.

As a left adjoint, T is right exact, so it remains to prove that T preserves monomorphisms. We proceed in several steps. Let $f\colon M \hookrightarrow N$ be a monomorphism of Λ-modules.

(a) Suppose that $N = \Lambda^{\oplus n} = H(G^{\oplus n})$ is finite free. Then we have a commutative diagram

$$\begin{tikzcd}
T(M) \arrow[r, "T(f)"] \arrow[dr] & TH(G^{\oplus n}) \arrow[d, "\varepsilon_{G^{\oplus n}}", "\sim"'] \\
& G^{\oplus n}.
\end{tikzcd}$$

By Step 2 the oblique arrow is a monomorphism, and the vertical map is an isomorphism. It follows that $T(f)$ is a monomorphism.

(b) Suppose that $N = \Lambda^{\oplus I}$ is free. Write $N = \varinjlim_{J\subseteq I} \Lambda^{\oplus J}$, where J runs through the finite subsets of I. Putting $M_J \coloneqq f^{-1}(\Lambda^{\oplus J})$, we can write $f\colon M \to N$ as the filtered colimit of maps $M_J \to \Lambda^{\oplus J}$. Since T commutes with colimits and filtered colimits in $\mathcal{A}$ are exact, it follows that

$$T(f) = \varinjlim_{\substack{J \subseteq I \\ \text{finite}}} T\big(M_J \to \Lambda^{\oplus J}\big)$$

is a monomorphism.

(c) The general case. Since $\mathsf{Mod}(\Lambda)$ has enough projectives, in order to prove that T is exact, it suffices to show that the first left derived functor $\mathrm{L}^1 T = 0$ vanishes (cf. the dual of Remark 14.6). So let $M \in \mathsf{Mod}(\Lambda)$ and choose a resolution $0 \to Q \to F \to M \to 0$ where F is free. We obtain an exact sequence

$$0 = \mathrm{L}^1 T(F) \to \mathrm{L}^1 T(M) \to T(Q) \to T(F) \to T(M) \to 0.$$

By (b) the map $T(Q) \hookrightarrow T(F)$ is a monomorphism, which implies $\mathrm{L}^1 T(M) = 0$ as desired.

This finishes the proof that T is exact. $\square$

Definition 24.7. Let $\mathcal{A}$ be an abelian category. An *essential extension* is a monomorphism $A \hookrightarrow E$ in $\mathcal{A}$ such that for every non-zero subobject $X \subseteq E$ it holds that $A \cap X \neq 0$.

As essential extension $A \hookrightarrow E$ is called an *injective hull* if E is an injective object in $\mathcal{A}$.

Theorem 24.8. *Let $\mathcal{A}$ be a Grothendieck category. Then $\mathcal{A}$ admits an injective cogenerator and every object admits an injective hull.*

Proof. **Step 1:** Ab has the injective cogenerator $\mathbb{Q}/\mathbb{Z}$.

It is well-known that an abelian group is injective if and only if it is divisible. Hence $\mathbb{Q}/\mathbb{Z}$ is injective. In order to show that $\mathbb{Q}/\mathbb{Z}$ is a cogenerator, it suffices to show the following: Let $A \in \mathsf{Ab}$ be an abelian group and $x \in A \setminus \{0\}$. Then there exists a homomorphism $f\colon A \to \mathbb{Q}/\mathbb{Z}$ with $f(x) \neq 0$. Indeed, let $\langle x \rangle \subseteq A$ be the subgroup generated by x. If $\langle x \rangle$ is finite of order n, we put $g\colon \langle x \rangle \to \mathbb{Q}/\mathbb{Z}$, $ix \mapsto \frac{i}{n}$. Otherwise, $\langle x \rangle \cong \mathbb{Z}$ and we put $g(ix) = \frac{i}{2}$. In any case we have $g(x) \neq 0$. Since $\mathbb{Q}/\mathbb{Z}$ is injective, g extends to a homomorphism $f\colon A \to \mathbb{Q}/\mathbb{Z}$ with $f(x) \neq 0$ as desired.

Step 2: $\mathsf{Mod}(\Lambda)$ (for a ring Λ) has the injective cogenerator $\mathrm{Hom}_{\mathsf{Ab}}(\Lambda, \mathbb{Q}/\mathbb{Z})$.

The forgetful functor $\mathsf{Mod}(\Lambda) \to \mathsf{Ab}$ is exact and admits a right adjoint $\mathrm{Hom}_{\mathsf{Ab}}(\Lambda, -)$. Hence, $\mathrm{Hom}_{\mathsf{Ab}}(\Lambda, \mathbb{Q}/\mathbb{Z})$ is injective. It is also a cogenerator, because if $M \in \mathsf{Mod}(\Lambda)$ injects into some $(\mathbb{Q}/\mathbb{Z})^I$, then we have a commutative diagram

$$\begin{tikzcd}
M \arrow[r] \arrow[dr, hook] & \mathrm{Hom}_{\mathsf{Ab}}(\Lambda, (\mathbb{Q}/\mathbb{Z})^I) \arrow[d, "\varphi \mapsto \varphi(1)"] = \mathrm{Hom}_{\mathsf{Ab}}(\Lambda, \mathbb{Q}/\mathbb{Z})^I \\
& (\mathbb{Q}/\mathbb{Z})^I.
\end{tikzcd}$$

It follows that the top horizontal map is a monomorphism as desired.

Step 3: We show that any Grothendieck category $\mathcal{A}$ with enough injectives admits injective hulls.

Fix $A \in \mathcal{A}$ and let $A \hookrightarrow I$ an embedding of A into an injective object I. The class $\mathcal{X}$ of essential extensions of A in I is non-empty and a partially ordered set (by Lemma 24.5) with respect to inclusion. If $(E_\alpha)_\alpha$ is a chain of essential extensions of A, then $E = \bigcup_\alpha E_\alpha$ is again an essential extension of A: Given any non-zero subobject $X \subseteq E$, we have $X = \bigcup_\alpha (E_\alpha \cap X)$ because $\mathcal{A}$ is AB5 (and hence filtered colimits commute with finite limits). Hence $E_\alpha \cap X \neq 0$ for some α, and then $A \cap X \neq 0$ because $A \hookrightarrow E_\alpha$ is essential. By Zorn's Lemma we find a maximal essential extension $A \hookrightarrow E$ inside I.

We now prove that E is a direct summand of I. By Zorn's lemma we find a maximal subobject $Q \subseteq I$ with $Q \cap E = 0$ (again, this uses that $\mathcal{A}$ is AB5). The composite $f\colon E \hookrightarrow I \xrightarrow{p} I/Q$ is a monomorphism by construction. It is also essential: If $X \subseteq I/Q$ is a subobject with $f^{-1}(X) = 0$, then $p^{-1}(X) \cap E = 0$. By maximality of Q we have $p^{-1}(X) = Q$, *i.e.*, $X = 0$. Since I is injective, there exists a map $g\colon I/Q \to I$ making the following diagram commute:

$$\begin{tikzcd} E \arrow[r, hook] \arrow[dr, hook] & I/Q \arrow[d, "g"] \\ & I. \end{tikzcd}$$

The image of g is an essential extension of E, hence an essential extension of A as is immediately verified. We deduce $\mathrm{Im}(g) = E$ by maximality of E. It follows that the essential extension $E \hookrightarrow I/Q$ splits; but this implies that it is an isomorphism. This shows $E \oplus Q = I$. In particular, E is an injective hull of A.

Step 4: There exists a ring Λ and functors

$$\mathcal{C} \underset{t}{\overset{i}{\rightleftarrows}} \mathsf{Mod}(\Lambda) \underset{H}{\overset{T}{\rightleftarrows}} \mathcal{A},$$

where:

(a) i is fully faithful and left adjoint to t,

(b) T is an exact Bousfield localization with fully faithful right adjoint H,

(c) $\mathcal{C} = \mathrm{Ker}(T)$,

(d) the composite $\mathcal{C}^\perp \to \mathsf{Mod}(\Lambda) \xrightarrow{T} \mathcal{A}$ is an equivalence, where
$$\mathcal{C}^\perp := \bigl\{ M \in \mathsf{Mod}(\Lambda) \bigm| \mathrm{Hom}(C, M) = 0 = \mathrm{Ext}^1(C, M) \text{ for all } C \in \mathcal{C} \bigr\} \subseteq \mathrm{Ker}(t).$$

(e) We have equalities $\mathsf{Inj}(\mathsf{Mod}(\Lambda)) \cap \mathrm{Ker}(t) = \mathsf{Inj}(\mathsf{Mod}(\Lambda)) \cap \mathcal{C}^\perp = \mathsf{Inj}(\mathcal{C}^\perp)$, where Inj denotes the class of injective objects. In particular, $\mathcal{C}^\perp$ is closed under injective hulls.

By Theorem 24.6 there exists a ring Λ and an exact Bousfield localization

$$T\colon \mathsf{Mod}(\Lambda) \to \mathcal{A}$$

with fully faithful right adjoint H. The inclusion $\mathcal{C} := \mathrm{Ker}(T) \hookrightarrow \mathsf{Mod}(\Lambda)$ admits a right adjoint t: Indeed, let $M \in \mathsf{Mod}(\Lambda)$ and denote by $\eta_M\colon M \to HT(M)$ the unit. Note that $T(\eta_M)$ is an isomorphism by Theorem 9.4(ii); in particular, $\mathrm{Ker}(\eta_M) \in \mathrm{Ker}(T)$. Now, for every $C \in \mathcal{C}$ we have an exact sequence

$$0 \to \mathrm{Hom}(C, \mathrm{Ker}(\eta_M)) \to \mathrm{Hom}(C, M) \xrightarrow{\eta_{M*}} \mathrm{Hom}(C, HT(M)) = \mathrm{Hom}(T(C), T(M)) = 0.$$

Hence, the right adjoint $t\colon \mathsf{Mod}(\Lambda) \to \mathcal{C}$ is given by $t(M) = \mathrm{Ker}(\eta_M)$.

We now prove part (d). By Theorem 9.4(iv) it suffices to identify $\mathcal{C}^\perp$ with the essential image of H, that is, the full subcategory of H-local objects. For $C \in \mathcal{C}$ and $A \in \mathcal{A}$ we have

$$\mathrm{Hom}(C, H(A)) = \mathrm{Hom}_{\mathcal{A}}(T(C), A) = 0.$$

Let now $0 \to H(A) \xrightarrow{f} E \to C \to 0$ be a short exact sequence. As T is exact and $T(C) = 0$, it follows that f is an H-local equivalence. In other words, precomposition with f induces an isomorphism $\mathrm{Hom}(E, H(A)) \xrightarrow{\sim} \mathrm{Hom}(H(A), H(A))$. We deduce that the short exact sequence splits, and hence $\mathrm{Ext}^1(C, H(A)) = 0$. Therefore, the H-local objects are contained in $\mathcal{C}^\perp$. Conversely, let $M \in \mathcal{C}^\perp$ and let $f\colon N \to N'$ be an H-local equivalence. As T is exact, this means $\mathrm{Ker}(f), \mathrm{Coker}(f) \in \mathcal{C}$. The exact sequences $0 \to \mathrm{Ker}(f) \to N \to \mathrm{Im}(f) \to 0$ and $0 \to \mathrm{Im}(f) \to N' \to \mathrm{Coker}(f) \to 0$ induce exact sequences

$$0 \to \mathrm{Hom}(\mathrm{Im}(f), M) \to \mathrm{Hom}(N, M) \to \mathrm{Hom}(\mathrm{Ker}(f), M) = 0$$

and

$$0 = \mathrm{Hom}(\mathrm{Coker}(f), M) \to \mathrm{Hom}(N', M) \to \mathrm{Hom}(\mathrm{Im}(f), M) \to \mathrm{Ext}^1(\mathrm{Coker}(f), M) = 0.$$

It follows that $f^*\colon \mathrm{Hom}(N', M) \xrightarrow{\sim} \mathrm{Hom}(\mathrm{Im}(f), M) \xrightarrow{\sim} \mathrm{Hom}(N, M)$ is an isomorphism, *i.e.*, M is H-local.

Finally, let us establish (e). The first equality is clear from the fact that $\mathrm{Ext}^1(-, I) = 0$ for any injective Λ-module I. By (d) the inclusion $\mathcal{C}^\perp \subseteq \mathsf{Mod}(\Lambda)$ is left exact and has an exact left adjoint. Therefore, the inclusion preserves injectives, we shows $\mathsf{Inj}(\mathcal{C}^\perp) \subseteq \mathsf{Inj}(\mathsf{Mod}(\Lambda)) \cap \mathcal{C}^\perp$. On the other hand, the left exactness visibly implies $\mathsf{Inj}(\mathsf{Mod}(\Lambda)) \cap \mathcal{C}^\perp \subseteq \mathsf{Inj}(\mathcal{C}^\perp)$.

It remains to check that $\mathcal{C}^\perp$ is closed under injective hulls. Let $A \in \mathcal{C}^\perp$ and let $A \to E(A)$ be the injective hull in $\mathsf{Mod}(\Lambda)$. Then $A \cap t(E(A)) = 0$, and hence $t(E(A)) = 0$ because $E(A)$ is an essential extension of A. Hence $E(A) \in \mathsf{Inj}(\mathsf{Mod}(\Lambda)) \cap \mathrm{Ker}(t) \subseteq \mathsf{Inj}(\mathcal{C}^\perp)$, by our previous observations. In particular, if A is injective in $\mathcal{C}^\perp$, then A is a retract of $E(A)$ and hence also injective in $\mathsf{Mod}(\Lambda)$.

Step 5: End of the proof.

By step 4, every object $A \in \mathcal{A}$ admits an injective hull, say $E(A)$. Moreover, $G = T(\Lambda)$ is a generator of $\mathcal{A}$, and the class $\mathcal{S}$ of subobjects of G forms a set by Lemma 24.5. We claim that $I \coloneqq \prod_{U \in \mathcal{S}} E(G/U)$ is an injective cogenerator of $\mathcal{A}$. It suffices to prove that every non-zero object $A \in \mathcal{A}$ admits a non-zero map $A \to I$. But this is clear because, since G is a generator, there is a non-zero map $G \to A$, which induces an embedding $G/U \hookrightarrow A$. But then the inclusion $G/U \hookrightarrow E(G/U) \subseteq I$ extends to a non-zero map $A \to I$ as desired. □

We will use the following deep fact without proof:

Fact 24.9. Let $\mathcal{A}$ be a Grothendieck category. Then $\mathsf{D}(\mathcal{A})$ is locally small.

Proof. This is proved in Theorem B.21. See also [Kra21, Proposition 4.3.8]. □

Theorem 24.10. *Let $\mathcal{A}$ be a Grothendieck category. Then the functor $Q\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is a Bousfield localization.*

Proof. Choose an exact Bousfield localization $T\colon \mathsf{Mod}(\Lambda) \to \mathcal{A}$, for some ring Λ with fully faithful right adjoint H (see Theorem 24.6). Then also $\mathsf{K}(T)\colon \mathsf{K}(\mathsf{Mod}(\Lambda)) \to \mathsf{K}(\mathcal{A})$ is a Bousfield localization with fully faithful right adjoint $\mathsf{K}(H)$. Note that $\mathsf{Mod}(\Lambda)$ is AB4* and admits an injective cogenerator by Theorem 24.8. By Theorem 23.8 the functor $Q_\Lambda\colon \mathsf{K}(\mathsf{Mod}(\Lambda)) \to \mathsf{D}(\mathsf{Mod}(\Lambda))$ is a Bousfield localization. Moreover, there is a derived functor

$$\mathsf{D}(T)\colon \mathsf{D}(\mathsf{Mod}(\Lambda)) \to \mathsf{D}(\mathcal{A}),$$

where T commutes with arbitrary direct sums, $\mathsf{D}(\mathcal{A})$ is locally small by Fact 24.9, and $\mathsf{D}(\mathsf{Mod}(\Lambda))$ is compactly generated by Example 23.7. Now Corollary 23.4 provides a right adjoint $\mathrm{R}H\colon \mathsf{D}(\mathcal{A}) \to \mathsf{D}(\mathsf{Mod}(\Lambda))$ of $\mathsf{D}(T)$. Consider now the following commutative diagram

$$\begin{tikzcd}
\mathsf{K}(\mathsf{Mod}(\Lambda)) \arrow[r, "Q_\Lambda"] \arrow[d, "\mathsf{K}(T)"'] & \mathsf{D}(\mathsf{Mod}(\Lambda)) \arrow[d, "\mathsf{D}(T)"] \\
\mathsf{K}(\mathcal{A}) \arrow[r, "Q_{\mathcal{A}}"] & \mathsf{D}(\mathcal{A})
\end{tikzcd}$$

in which $\mathsf{K}(T)$ is a Bousfield localization and $\mathsf{D}(T)Q_\Lambda\colon \mathsf{K}(\mathsf{Mod}(\Lambda)) \to \mathsf{D}(\mathcal{A})$ admits a right adjoint. Applying Lemma 8.8 with $(\mathcal{C}, S, \mathcal{D}, T) = (\mathsf{K}(\mathsf{Mod}(\Lambda)), \mathsf{K}(T)^{-1}(\{iso\}), \mathsf{D}(\mathcal{A}), \{iso\})$ shows that the localization functor $Q_{\mathcal{A}}\colon \mathsf{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ admits a right adjoint (which then is necessarily fully faithful by Exercise 8.11). Hence, $Q_{\mathcal{A}}$ is a Bousfield localization. □

B. Presentable categories

The goal of this section is to give a proof of Fact 24.9. This will be achieved in Theorem B.21.

Definition B.1. A cardinal κ is called *regular* if for every collection $(\lambda_i)_{i\in I}$ of cardinals we have $\sum_{i\in I} \lambda_i < \kappa$ provided that $|I| < \kappa$ and $\lambda_i < \kappa$ for all $i \in I$.

Example B.2. (i) $\aleph_0$ is regular.

(ii) The successor κ^+ of an infinite cardinal κ is regular. In particular, there exist infinitely many regular cardinals.

Definition B.3. Let κ be a regular cardinal. A category $\mathcal{I}$ is called *κ-filtered* if:

- $\mathcal{I} \neq \varnothing$;
- for any family of objects $(x_i)_{i\in I}$ with $|I| < \kappa$ there exists $x \in \mathcal{I}$ and morphisms $x_i \to x$, for all i;
- for any family of morphisms $(f_i\colon x \to y)_{i\in I}$ with $|I| < \kappa$ there exists a morphism $f\colon y \to z$ such that $f \circ f_i = f \circ f_j$ for all i, j.

The colimit over a κ-filtered diagram will usually be denoted by "$\varinjlim$" instead of "colim".

Example B.4. (i) A $\aleph_0$-filtered category is the same as a filtered category.

(ii) If $\lambda \leq \kappa$ are regular cardinals, then every κ-filtered category is also λ-filtered.

(iii) The category Set^κ of κ-small sets is κ-filtered, for any regular cardinal κ.

(iv) If κ is a regular cardinal and $\mathcal{C}$ is a category admitting κ-small colimits, then $\mathcal{C}$ is κ-filtered.

An important property of the category Set is that κ-filtered colimits commute with κ-small limits:

Lemma B.5. *Let κ be a regular cardinal. Let $\mathcal{I}$ be a small κ-filtered category and $\mathcal{J}$ a κ-small category. Then for every functor $F\colon \mathcal{I} \times \mathcal{J} \to \mathsf{Set}$ the canonical map*

$$\varinjlim_{i\in\mathcal{I}} \lim_{j\in\mathcal{J}} F(i,j) \xrightarrow{\sim} \lim_{j\in\mathcal{J}} \varinjlim_{i\in\mathcal{I}} F(i,j)$$

is an isomorphism.

Proof sketch. An element of the right hand side consists of a function $\varphi\colon \mathrm{Ob}(\mathcal{J}) \to \mathrm{Ob}(\mathcal{I})$ and objects $x_j \in F(\varphi(j), j)$ for all $j \in J$ which are compatible under the transition maps. As $\mathcal{I}$ is κ-filtered and $\mathcal{J}$ is κ-small, there exists $i \in \mathcal{I}$ and maps $f_j\colon \varphi(j) \to i$ for all $j \in \mathcal{J}$. Then the image of the x_j in $F(i,j)$ assemble into an element of the left hand side, which shows that the map is surjective.

For injectivity, let $x, y \in \varinjlim_i \lim_j F(i,j)$ whose images in $\lim_j \varinjlim_i F(i,j)$ agree. Fix $i_0 \in \mathcal{I}$ such that $x, y \in \lim_j F(i_0, j)$. Then for each $j \in \mathcal{J}$ there exists $f_j\colon i_0 \to i_j$ such that $x_j = y_j$ in $F(i_j, j)$. As $\mathcal{I}$ is κ-filtered and $\mathcal{J}$ is κ-small, we find $i_1 \in \mathcal{I}$ and maps $g_j\colon i_j \to i_1$ in $\mathcal{I}$ such that $g_j f_j = g_{j'} f_{j'}$ for all $j, j' \in \mathcal{J}$. But then $x = y$ in $\lim_j F(i_2, j)$, which proves injectivity. $\square$

Lemma B.6. *Let κ be a regular cardinal.*

(i) Every small category $\mathcal{I}$ is a κ-filtered union of κ-small categories.

(ii) Let $\mathcal{C}$ be a category and $X\colon \mathcal{I} \to \mathcal{C}$ a diagram. Suppose that $\mathcal{I} = \bigcup_{j\in\mathcal{J}} \mathcal{I}_j$ is a κ-filtered union and that the colimit $\mathrm{colim}_{i\in\mathcal{I}_j} X_i$ exists in $\mathcal{C}$, for all $j \in \mathcal{J}$. Then the canonical map

$$\varinjlim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}_j} X_i \xrightarrow{\sim} \operatorname*{colim}_{i\in\mathcal{I}} X_i$$

is an isomorphism (that is, if either colimit exists, then so does the other, and the map is an isomorphism)

Proof. For part (i), let $\binom{\mathcal{I}}{\kappa}$ be the partially ordered set of all κ-small subcategories of $\mathcal{I}$. Then $\binom{\mathcal{I}}{\kappa}$ is κ-filtered and $\mathcal{I} = \bigcup_{\mathcal{J}\in\binom{\mathcal{I}}{\kappa}} \mathcal{J}$ is a κ-filtered colimit of κ-small categories.

Part (ii) follows easily by comparing the universal properties. $\square$

Definition B.7. Let κ be a regular cardinal and $\mathcal{C}$ a cocomplete category. An object $X \in \mathcal{C}$ is called *κ-compact* if the functor

$$\mathrm{Hom}_{\mathcal{C}}(X, -)\colon \mathcal{C} \to \mathsf{Set}$$

commutes with κ-filtered colimits. We denote by $\mathcal{C}^\kappa \subseteq \mathcal{C}$ the full subcategory of κ-compact objects.

Example B.8. (i) If Λ is a ring, then a Λ-module is compact (*i.e.*, $\aleph_0$-compact) if and only if it is finitely presented.

(ii) If $\lambda \le \kappa$ are regular cardinals, then every λ-compact object is also κ-compact: $\mathcal{C}^\lambda \subseteq \mathcal{C}^\kappa$.

Lemma B.9. *Let κ be a regular cardinal and $\mathcal{C}$ a cocomplete category. Then $\mathcal{C}^\kappa$ is closed under κ-small colimits.*

Proof. Let $(X_j)_{j\in\mathcal{J}}$ be a κ-small diagram in $\mathcal{C}^\kappa$. For any κ-filtered diagram $(Y_i)_{i\in\mathcal{I}}$ in $\mathcal{C}$ we compute

$$\begin{aligned}\varinjlim_{i\in\mathcal{I}} \operatorname{Hom}\bigl(\operatorname*{colim}_{j\in\mathcal{J}} X_j, Y_i\bigr) &\cong \varinjlim_i \lim_j \operatorname{Hom}(X_j, Y_i) \\ &\cong \lim_j \varinjlim_i \operatorname{Hom}(X_j, Y_i) && \text{(Lemma B.5)} \\ &\cong \lim_j \operatorname{Hom}(X_j, \varinjlim_i Y_i) && (X_j \in \mathcal{C}^\kappa) \\ &\cong \operatorname{Hom}(\operatorname*{colim}_j X_j, \varinjlim_i Y_i),\end{aligned}$$

which shows $\operatorname{colim}_j X_j \in \mathcal{C}^\kappa$ as desired. □

Lemma B.10. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction and κ a regular cardinal. If U preserves κ-filtered colimits, then $F(\mathcal{C}^\kappa) \subseteq \mathcal{D}^\kappa$.*

Proof. Let $C \in \mathcal{C}^\kappa$ be κ-compact and $D = \varinjlim_{i\in\mathcal{I}} D_i$ a κ-filtered colimit in D. Then

$$\begin{aligned}\varinjlim_{i\in\mathcal{I}} \operatorname{Hom}(F(C), D_i) &= \varinjlim_{i\in\mathcal{I}} \operatorname{Hom}(C, U(D_i)) \\ &= \operatorname{Hom}\bigl(C, \varinjlim_{i\in\mathcal{I}} U(D_i)\bigr) \\ &= \operatorname{Hom}\bigl(C, U(\varinjlim_{i\in\mathcal{I}} D_i)\bigr) \\ &= \operatorname{Hom}\bigl(F(C), \varinjlim_{i\in\mathcal{I}} D_i\bigr),\end{aligned}$$

which shows that $F(C)$ is κ-compact. □

Definition B.11. Let κ be a regular cardinal. A category $\mathcal{C}$ is called *κ-presentable* if it is cocomplete, $\mathcal{C}^\kappa$ is (essentially) small and every object is a κ-filtered colimit of κ-compact objects.

The category $\mathcal{C}$ is called *presentable* if it is κ-presentable for some κ.

Proposition B.12. *Let $\mathcal{C}$ be a κ-presentable category, for some regular cardinal κ.*

(i) $\mathcal{C}$ is λ-presentable, for every regular cardinal $\lambda \geq \kappa$.

(ii) Let $\lambda \geq \kappa$ be a regular cardinal. Then every λ-compact object is a λ-small, κ-filtered colimit of κ-compact objects.

(iii) $\mathcal{C} = \bigcup_\lambda \mathcal{C}^\lambda$, where λ runs through the regular cardinals. In particular, every small set of objects of $\mathcal{C}$ is contained in $\mathcal{C}^\kappa$, for some regular cardinal κ.

Proof. Let $X = \varinjlim_{i\in\mathcal{I}} X_i$ be a κ-filtered colimit of κ-compact objects, and let $\lambda \geq \kappa$ be a regular cardinal. By Lemma B.6 we also have

$$X = \varinjlim_{j\in\mathcal{J}} \operatorname*{colim}_{i\in\mathcal{I}_j} X_i,$$

where $\mathcal{J}$ is λ-filtered and each $\mathcal{I}_j$ is λ-small. As each $\mathrm{colim}_{i\in\mathcal{I}_j} X_i$ is λ-compact by Lemma B.9, we deduce part (i). If moreover we assume that X is λ-compact, then the identity $X \to X$ factors through $\mathrm{colim}_{i\in\mathcal{I}_j} X_i$ for some $j \in \mathcal{J}$. Hence, X is a retract of $\mathrm{colim}_{i\in\mathcal{I}_j} X_i$ and therefore itself a λ-small colimit of κ-compact objects. This proves (ii). Finally, for part (iii) we pick some regular cardinal $\mu \geq \kappa$ with $|\mathcal{I}| < \mu$. Then $X = \mathrm{colim}_{i\in\mathcal{I}} X_i$ is μ-compact by Lemma B.9. □

Example B.13. The categories Set, Ab and $\mathsf{Mod}(\Lambda)$, for a ring Λ, are $\aleph_0$-presentable.

The category Top of topological spaces is not presentable.

Lemma B.14. *Let κ be a regular cardinal and $\mathcal{C}$ a κ-presentable category.*

Then every morphism $f\colon X \to Y$ is of the form

$$\varinjlim_{i\in\mathcal{I}} \varphi_i\colon \varinjlim_{i\in\mathcal{I}} X_i \to \varinjlim_{i\in\mathcal{I}} Y_i,$$

where $\mathcal{I}$ is a κ-filtered category. Moreover, if $\lambda \geq \kappa$ is a regular cardinal such that $\mathcal{C}^\kappa$ is λ-small and $X, Y \in \mathcal{C}^\lambda$, then the category $\mathcal{I}$ can be chosen to be λ-small.

Proof. Let $f\colon X \to Y$ a map in $\mathcal{C}$, and choose a regular cardinal $\lambda \geq \kappa$ such that $\mathcal{C}^\kappa$ is λ-small and $X, Y \in \mathcal{C}^\lambda$ (it exists by Proposition B.12). We write $X = \varinjlim_{i\in\mathcal{I}} X_i$ and $Y = \varinjlim_{j\in\mathcal{J}} Y_j$ as λ-small and κ-filtered colimits with $X_i, Y_j \in \mathcal{C}^\kappa$. Consider the category

$$\mathcal{K} \coloneqq \mathcal{I} \times_{\mathcal{C}^\kappa/Y} \mathrm{Fun}([1], \mathcal{C}^\kappa/Y) \times_{\mathcal{C}^\kappa/Y} \mathcal{J}.$$

The objects are triples (i, j, ϕ) consisting of objects $i \in \mathcal{I}$, $j \in \mathcal{J}$ and a morphism $\phi\colon X_i \to Y_j$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\phi"] \arrow[d] & Y_j \arrow[d] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

commutes. A morphism $(i, j, \phi) \to (i', j', \phi')$ is a pair (α, β) consisting of maps $\alpha\colon i \to i'$ and $\beta\colon j \to j'$ such that the diagram

$$\begin{tikzcd} X_i \arrow[r, "\phi"] \arrow[d, "X_\alpha"'] & Y_j \arrow[d, "Y_\beta"] \\ X_{i'} \arrow[r, "\phi'"'] & Y_{j'} \end{tikzcd}$$

commutes. We make the following claims:

(i) $\mathcal{K}$ is λ-small and κ-filtered.

(ii) The projection maps $p_{\mathcal{I}}\colon \mathcal{K} \to \mathcal{I}$ and $p_{\mathcal{J}}\colon \mathcal{K} \to \mathcal{J}$ are cofinal.

(iii) Let $\varphi\colon X_\bullet \circ p_{\mathcal{I}} \to Y_\bullet \circ p_{\mathcal{J}}$ be the natural transformation of functors $\mathcal{K} \to \mathcal{C}^\kappa/Y$ given by the projection $\mathcal{K} \to \mathrm{Fun}([1], \mathcal{C}^\kappa/Y)$. Then the following diagram commutes:

$$\begin{tikzcd} \varinjlim_{k\in\mathcal{K}} X_{p_{\mathcal{I}}(k)} \arrow[r, "\varinjlim \varphi"] \arrow[d, "\simeq"'] & \varinjlim_{k\in\mathcal{K}} Y_{p_{\mathcal{J}}(k)} \arrow[d, "\simeq"] \\ X \arrow[r, "f"'] & Y. \end{tikzcd}$$

We first show (i). Since $\mathcal{C}^\kappa$, $\mathcal{I}$ and $\mathcal{J}$ are λ-small, it is clear that $\mathcal{K}$ is κ-small and non-empty. Let now $(k_n)_{n\in N}$ in $\mathcal{K}$ with $|N| < \kappa$. As $\mathcal{I}$ is κ-filtered, there exists $i \in \mathcal{I}$ together with maps $p_{\mathcal{I}}(k_n) \to i$ for all $n \in N$. As X_i is κ-compact, the composite $X_i \to X \to Y = \varinjlim_{j\in\mathcal{J}} Y_j$ factors through some Y_{j_0}. As $\mathcal{J}$ is κ-filtered, there exists $j \in \mathcal{J}$ and maps $p_{\mathcal{J}}(k_n) \to j$ and $j_0 \to j$. Denoting by $\phi\colon X_i \to Y_j$ the induced map, it follows that (i, j, ϕ) is an upper bound for $(k_n)_n$. Similarly, let $(f_n\colon k \to k')_n$ be a κ-small family of maps in $\mathcal{K}$. As $\mathcal{I}$ is κ-filtered, we find a map $\alpha\colon p_{\mathcal{I}}(k') \to i$ such that $\alpha \circ p_{\mathcal{I}}(f_n) = \alpha \circ p_{\mathcal{I}}(f_m)$ for all n, m. Again, let $j_0 \in \mathcal{J}$ such that the composite $X_i \to X \to Y$ factors through Y_{j_0}. As $\mathcal{J}$ is κ-filtered, there exists $j \in \mathcal{J}$ and maps $j_0 \to j$ and $\beta\colon p_{\mathcal{J}}(k') \to j$ such that $\beta \circ p_{\mathcal{J}}(f_n) = \beta \circ p_{\mathcal{J}}(f_m)$ for all n, m. Denoting by $\phi\colon X_i \to Y_j$ the induced map, we obtain a map $f\colon k' \to (i, j, \phi)$ such that $f \circ f_n = f \circ f_m$ for all n, m. Hence, $\mathcal{K}$ is κ-filtered.

We now prove (ii). Since $\mathcal{I}$ is filtered, it suffices to show that $p_{\mathcal{I}}\colon \mathcal{K} \to \mathcal{I}$ is surjective on objects. But this is clear: Let $i \in \mathcal{I}$. As X_i is κ-compact and $\mathcal{J}$ is κ-filtered, the composite $X_i \to X \to Y = \varinjlim_j Y_j$ factors through a map $\phi\colon X_i \to Y_j$, and hence $(i, j, \phi) \in \mathcal{K}$ is a preimage of i. Let now $j_0 \in \mathcal{J}$ and $k_1 = (i_1, j_1, \phi_1) \in \mathcal{K}$ arbitrary. As $\mathcal{J}$ is filtered, there exists $j \in \mathcal{J}$ and maps $\beta\colon j_1 \to j$ and $j_0 \to j$. Now, $k = (i_1, j, Y_\beta\phi_1) \in \mathcal{K}$ is an object such that $j_0 \to j = p_{\mathcal{J}}(k)$. As $\mathcal{J}$ is filtered, this implies that $p_{\mathcal{J}}$ is cofinal.

The vertical maps in (iii) are isomorphisms by (ii), and the commutativity is clear. □

Lemma B.15. *Let Λ be a ring and κ a regular cardinal. Let $M \in \mathsf{Mod}(\Lambda)$ be a module with presentation*

$$\Lambda^{(\lambda_{n+1})} \to \cdots \to \Lambda^{(\lambda_0)} \to M \to 0,$$

where $\lambda_0, \ldots, \lambda_{n+1} < \kappa$. Then $\operatorname{Ext}^n(M, -)\colon \mathsf{Mod}(\Lambda) \to \mathsf{Ab}$ commutes with κ-filtered colimits.

Proof. Let $X = \varinjlim_{i\in\mathcal{I}} X_i$ be a κ-filtered colimit of Λ-modules. Then

$$\begin{aligned}
\varinjlim_{i\in\mathcal{I}} \operatorname{Ext}^n(M, X_i) &= \varinjlim_{i\in\mathcal{I}} \mathrm{H}^n \operatorname{Hom}(\Lambda^{(\lambda_\bullet)}, X_i) \\
&= \mathrm{H}^n\big(\varinjlim_{i\in\mathcal{I}} \operatorname{Hom}(\Lambda^{(\lambda_\bullet)}, X_i)\big) \\
&= \mathrm{H}^n \operatorname{Hom}\big(\Lambda^{(\lambda_\bullet)}, \varinjlim_{i\in\mathcal{I}} X_i\big) \\
&= \operatorname{Ext}^n\big(M, \varinjlim_{i\in\mathcal{I}} X_i\big),
\end{aligned}$$

where the second identity uses that filtered colimits in $\mathsf{Mod}(\Lambda)$ are exact. □

Lemma B.16. *Let $X \twoheadrightarrow Y$ be an epimorphism in a Grothendieck category $\mathcal{A}$. Suppose that Y is κ-compact and that $X = \varinjlim_{i\in\mathcal{I}} X_i$ is a κ-filtered colimit. Then there exists $i \in \mathcal{I}$ such that the composite $X_i \to X \to Y$ is an epimorphism.*

Proof. Denote by Y_i' the image of $X_i \to X \to Y$. We obtain a factorization

$$X = \varinjlim_i X_i \twoheadrightarrow \varinjlim_i Y_i' \hookrightarrow Y,$$

because the formation of filtered colimits is exact in $\mathcal{A}$. As $X \twoheadrightarrow Y$ is an epimorphism, we deduce that $\varinjlim_i Y_i' = Y$. As Y is κ-compact, the identity $Y \to Y$ factors through some Y_i', from which we deduce $Y_i' = Y$ as desired. □

Theorem B.17. *Every Grothendieck category $\mathcal{A}$ is presentable.*

Proof. By step 4 in the proof of Theorem 24.8 there exists a ring Λ and functors

$$\operatorname{Ker}(T) \underset{t}{\overset{i}{\rightleftarrows}} \mathsf{Mod}(\Lambda) \underset{H}{\overset{T}{\rightleftarrows}} \mathcal{A},$$

where T is a Bousfield localization with fully faithful right adjoint H and t is a right adjoint of i. Moreover, we have an equivalence of categories

$$\mathcal{A} \cong \operatorname{Ker}(T)^{\perp} = \left\{ M \in \mathsf{Mod}(\Lambda) \,\middle|\, \operatorname{Hom}(C, M) = 0 = \operatorname{Ext}^1(C, M) \text{ for all } C \in \operatorname{Ker}(T) \right\}.$$

Observe that $K = \bigoplus_{U \in \mathcal{S}} \Lambda/U$ is a generator of $\operatorname{Ker}(T)$, where $\mathcal{S}$ is the set of all submodules $U \subseteq \Lambda$ with $\Lambda/U \in \operatorname{Ker}(T)$ (*e.g.*, by checking the criterion (d) in Lemma 24.3). We claim that

$$\operatorname{Ker}(T)^{\perp} = K^{\perp}.$$

The inclusion "$\subseteq$" is trivial. Conversely, let $M \in K^{\perp}$. For any $X \in \operatorname{Ker}(T)$ we choose a presentation $0 \to Y \to K^{(\lambda)} \to X \to 0$ for some cardinal λ. Note that $Y \in \operatorname{Ker}(T)$. The exact sequence $0 \to \operatorname{Hom}(X, M) \to \operatorname{Hom}(K^{(\lambda)}, M) = 0$ shows $\operatorname{Hom}(X, M) = 0$. As X was arbitrary, it also holds that $\operatorname{Hom}(Y, M) = 0$. Keeping in mind that $\operatorname{Ext}^1(K^{(\lambda)}, M) = \operatorname{Ext}^1(K, M)^{\lambda} = 0$, we deduce from the exact sequence $0 = \operatorname{Hom}(Y, M) \to \operatorname{Ext}^1(X, M) \to \operatorname{Ext}^1(K^{(\lambda)}, M) = 0$ that $\operatorname{Ext}^1(X, M) = 0$, and hence $M \in \operatorname{Ker}(T)^{\perp}$.

Now choose a free presentation $\Lambda^{(\lambda_2)} \to \Lambda^{(\lambda_1)} \to \Lambda^{(\lambda_0)} \to K \to 0$ and a regular cardinal $\kappa \geq \lambda_2, \lambda_1, \lambda_0$. Then $\operatorname{Hom}(K, -)$ and $\operatorname{Ext}^1(K, -)$ preserve κ-filtered colimits by Lemma B.15. Hence $\operatorname{Ker}(T)^{\perp}$ is closed under κ-filtered colimits. In other words: The functor $H\colon \mathcal{A} \to \mathsf{Mod}(\Lambda)$ preserves κ-filtered colimits. But then Lemma B.10 shows that T preserves κ-compact objects.

Now, every module is a filtered colimit of finitely presented modules and hence also a κ-filtered colimit of κ-compact objects, by Proposition B.12. Applying T then shows that every object of $\mathcal{A}$ is a κ-filtered colimit of κ-compact objects. Therefore, $\mathcal{A}$ is κ-presentable. $\square$

Lemma B.18. *Let $\mathcal{A}$ be a Grothendieck category and κ a regular cardinal such that $\mathcal{A}$ is κ-presentable. Then $\mathcal{A}^{\kappa}$ is abelian if and only if $\mathcal{A}^{\kappa}$ is closed under kernels. In this case, the inclusion $\mathcal{A}^{\kappa} \hookrightarrow \mathcal{A}$ is exact and closed under extensions.*

Proof. Observe that $\mathcal{A}^{\kappa}$ is closed under cokernels. Hence, if $\mathcal{A}^{\kappa}$ is closed under kernels, then $\mathcal{A}^{\kappa}$ is an abelian subcategory. Conversely, suppose that $\mathcal{A}^{\kappa}$ is abelian. Let $0 \to X \to Y \xrightarrow{f} Z$ be an exact sequence in $\mathcal{A}^{\kappa}$. We need to show that X is also the kernel of f in $\mathcal{A}$. Let $A \in \mathcal{A}$ and write $A = \varinjlim_{i \in \mathcal{I}} A_i$ as a κ-filtered colimit of κ-compact objects. Consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \lim_{i \in \mathcal{I}^{\mathrm{op}}} \operatorname{Hom}(A_i, X) \arrow[r] \arrow[d, equal] & \lim_{i \in \mathcal{I}^{\mathrm{op}}} \operatorname{Hom}(A_i, Y) \arrow[r] \arrow[d, equal] & \lim_{i \in \mathcal{I}^{\mathrm{op}}} \operatorname{Hom}(A_i, Z) \arrow[d, equal] \\
0 \arrow[r] & \operatorname{Hom}(A, X) \arrow[r] & \operatorname{Hom}(A, Y) \arrow[r] & \operatorname{Hom}(A, Z),
\end{tikzcd}$$

where the first row is exact. It follows that the second row is exact. In other words, X is also the kernel of f in $\mathcal{A}$.

Suppose now that $\mathcal{A}^{\kappa}$ is abelian. It remains to prove that $\mathcal{A}^{\kappa}$ is closed under extensions. Let $E \coloneqq [0 \to X \to Y \to Z \to 0]$ be an extension in $\mathcal{A}$ with $X, Z \in \mathcal{A}^{\kappa}$. Write $Y = \varinjlim_{i \in \mathcal{I}} Y_i$

as a κ-filtered colimit of κ-compact objects, so that E is a κ-filtered colimit of exact sequences $0 \to X_i \to Y_i \to Z$. By Lemma B.16 there exists $i_0 \in \mathcal{I}$ such that $Y_{i_0} \twoheadrightarrow Z$ is an epimorphism. Observe that $X = \varinjlim_{i\in\mathcal{I}} X_i$, and hence the κ-compactness of X implies that $X \xrightarrow{\sim} \varinjlim_{i\in\mathcal{I}} X_i$ factors through X_{i_1}, for some $i_1 \in \mathcal{I}$. Hence, the map $g\colon X_{i_1} \twoheadrightarrow X$ is an epimorphism. Enlarging i_0 and i_1, if necessary, we may assume $i_0 = i_1$. Now, we have a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Ker}(g) \arrow[r, equal] \arrow[d] & \mathrm{Ker}(g) \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_{i_1} \arrow[r] \arrow[d, two heads, "g"'] & Y_{i_1} \arrow[r] \arrow[d] & Z \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & Y \arrow[r] & Z \arrow[r] & 0
\end{tikzcd}$$

with exact rows. By the four lemma, we deduce that $Y_{i_1} \twoheadrightarrow Y$ is an isomorphism with kernel $\mathrm{Ker}(g)$. As $\mathrm{Ker}(g)$ is κ-compact by assumption, we conclude that $Y = \mathrm{Coker}(\mathrm{Ker}(g) \to Y_{i_1})$ is κ-compact as desired. □

Proposition B.19. *Let $\mathcal{A}$ be a Grothendieck category. There exists a regular cardinal λ_0 such that for all regular cardinals $\lambda \geq \lambda_0$ the subcategory $\mathcal{A}^\lambda \subseteq \mathcal{A}$ is abelian and closed under extensions.*

Proof. Let κ be a regular cardinal such that $\mathcal{A}$ is κ-presentable. Choose the regular cardinal $\lambda_0 \geq \kappa$ such that $\mathcal{A}^\kappa$ is λ_0-small and the kernel of each map in $\mathcal{A}^\kappa$ is λ_0-compact.

For all regular cardinals $\lambda \geq \lambda_0$ we claim that $\mathcal{A}^\lambda$ is abelian and closed under extensions. By Lemma B.18 it suffices to check that $\mathcal{A}^\lambda$ is closed under kernels.

Let $f\colon X \to Y$ be a morphism in $\mathcal{A}^\lambda$. By Lemma B.14 we may suppose that f is of the form

$$\varinjlim_{i\in\mathcal{I}} f_i\colon \varinjlim_{i\in\mathcal{I}} X_i \to \varinjlim_{i\in\mathcal{I}} Y_i,$$

where $X_i, Y_i \in \mathcal{A}^\kappa$ and $\mathcal{I}$ is a λ-small and κ-filtered category. By assumption, $\mathrm{Ker}(f_i)$ is λ-compact. As $\mathcal{A}$ is AB5, we deduce from Lemma B.9 that $\mathrm{Ker}(f) = \varinjlim_{i\in\mathcal{I}} \mathrm{Ker}(f_i)$ is λ-compact as desired. □

Lemma B.20. *Let $\kappa > \aleph_0$ be a regular cardinal and $\mathcal{A}$ a κ-presentable Grothendieck category such that $\mathcal{A}^\kappa$ is abelian. Then a morphism $X \to Y$ in $\mathsf{C}(\mathcal{A})$ with $X \in \mathsf{C}(\mathcal{A}^\kappa)$ and Y acyclic factors through an acyclic object in $\mathsf{C}(\mathcal{A}^\kappa)$.*

Proof. We need $\kappa > \aleph_0$ to ensure that $\mathsf{C}(\mathcal{A}^\kappa)$ is closed under countable colimits.

Suppose first that $X \in \mathsf{C}^-(\mathcal{A}^\kappa)$. Let $a \in \mathbb{Z}$ such that $\mathrm{H}^i(X) = 0$ for all $i > a$. We will inductively construct a sequence

$$X = X_a \to X_{a-1} \to X_{a-2} \to \cdots \to Y$$

in $\mathsf{C}(\mathcal{A}^\kappa)$ such that $\mathrm{H}^i(X_n) = 0$ for all $i > n$. Suppose that X_n has been defined. Consider the pullback

$$\begin{tikzcd}
V \arrow[r, two heads] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \mathrm{Ker}(d_{X_n}\colon X_n^n \to X_n^{n+1}) \arrow[d] \\
Y^{n-1} \arrow[r, two heads] & \mathrm{Ker}(d_Y\colon Y^n \to Y^{n+1})
\end{tikzcd}$$

and write $V = \varinjlim_i V_i$ as a κ-filtered colimit with $V_i \in \mathcal{A}^\kappa$. Observe that the top horizontal arrow is an epimorphism by Proposition 3.10. As X_n^{n-1} and $\operatorname{Ker}(d_{X_n})$ are κ-compact, it follows from Lemma B.16 that the map $X_n^{n-1} \to V$ factors through some V_j such that the composite $V_j \to V \to \operatorname{Ker}(d_{X_n})$ is an epimorphism. Now define a complex X_{n-1} by $X_{n-1}^i = X_n^i$ if $i \neq n-1$ and $X_{n-1}^{n-1} = V_j$, with the differential induced by the one on X_n. By construction we have $\mathrm{H}^i(X_{n-1}) = 0$ for all $i > n-1$ and $X_{n-1} \in \mathsf{C}(\mathcal{A}^\kappa)$. Moreover, $X' \coloneqq \varinjlim_n X_n$ is an acyclic complex in $\mathsf{C}(\mathcal{A}^\kappa)$ such that $X \to Y$ factors through X'.

Let now $X \in \mathsf{C}(\mathcal{A}^\kappa)$ be general. Then $X = \varinjlim_{n\geq 0} \tau^{\leq n} X$. Suppose that for some $n \geq 0$ we have defined an acyclic complex X_n in $\mathsf{C}(\mathcal{A}^\kappa)$ such that $\tau^{\leq n} X \to Y$ factors through X_n. Consider the pushout

$$\begin{tikzcd} \tau^{\leq n} X \arrow[r] \arrow[d] & \tau^{\leq n+1} X \arrow[d] \\ X_n \arrow[r] \arrow[ur, phantom, "\ulcorner"] & X'_{n+1} \end{tikzcd}$$

and observe that X'_{n+1} lies in $\mathsf{C}^-(\mathcal{A}^\kappa)$. By the definition of the pushout, the map $X_n \to Y$ factors through a map $X'_{n+1} \to Y$, which by the discussion above factors through some acyclic complex $X_{n+1} \in \mathsf{C}^-(\mathcal{A}^\kappa)$. By the construction the diagram

$$\begin{tikzcd} \tau^{\leq n} X \arrow[r] \arrow[d] & \tau^{\leq n+1} X \arrow[d] \\ X_n \arrow[r] & X_{n+1} \end{tikzcd}$$

commutes. Passing to the colimit, we obtain a factorization $X = \varinjlim_n \tau^{\leq n} X \to \varinjlim_n X_n \to Y$ where $\varinjlim_n X_n \in \mathsf{C}(\mathcal{A}^\kappa)$ is acyclic, since the formation of filtered colimits in $\mathcal{A}$ is exact. □

Theorem B.21. *Let $\mathcal{A}$ be a κ_0-presentable Grothendieck category, where $\kappa_0 > \aleph_0$ is a regular cardinal such that $\mathcal{A}^{\kappa_0}$ is abelian. For every regular cardinal $\kappa \geq \kappa_0$ the inclusion $\mathcal{A}^\kappa \subseteq \mathcal{A}$ induces a fully faithful functor $\mathsf{D}(\mathcal{A}^\kappa) \to \mathsf{D}(\mathcal{A})$. In particular,*

$$\mathsf{D}(\mathcal{A}) = \bigcup_{\kappa \geq \kappa_0} \mathsf{D}(\mathcal{A}^\kappa)$$

is a locally small category.

Proof. From Lemma B.20 and Exercise 11.5 we deduce that the inclusion $\mathsf{K}(\mathcal{A}^\kappa) \hookrightarrow \mathsf{K}(\mathcal{A})$ induces a fully faithful functor $\mathsf{D}(\mathcal{A}^\kappa) \to \mathsf{D}(\mathcal{A})$. Finally observe that every complex belongs to $\mathcal{A}^\kappa$ for some $\kappa \geq \kappa_0$ by Proposition B.12(iii). □

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